UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

22 April 2015, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***three*** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 A spin-1/2 evolves according to the time-dependent Hamiltonian

$$H(t) = \hbar\Omega S^z + \hbar\Phi S^x \sum_{n=0}^{\infty}\delta(t - nT),$$

where $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$.

(a) Show that the evolution operator from just before the first 'pulse' to just before the $(N+1)^{\text{th}}$, is given by

$$U(NT - \epsilon, -\epsilon) = (U_\Omega U_\Phi)^N,$$

where

$$U_\Omega = \begin{pmatrix} e^{-i\Omega T/2} & 0 \\ 0 & e^{i\Omega T/2} \end{pmatrix}, \qquad U_\Phi = \begin{pmatrix} \cos\Phi/2 & -i\sin\Phi/2 \\ -i\sin\Phi/2 & \cos\Phi/2 \end{pmatrix}.$$ [7]

(b) Show that, if $\boldsymbol{\theta} = \theta\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector

$$\exp\left(-i\boldsymbol{\theta}\cdot\mathbf{S}\right) = \mathbb{1}\cos\left(\theta/2\right) - i\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\sin\left(\theta/2\right).$$ [7]

(c) Apply the result of (b) to the evolution operator in (a) to find the angle through which the spin rotates, and its axis of rotation. [12]

(d) How would the answer to part (c) change if we were to consider a spin S instead? Explain your reasoning. [7]

Solution 1. (a) State that overall evolution is obtained by compunding a cycle consisting of a pulse with evolution U_Φ followed by precession with evolution operator U_Ω. $U_\Omega = e^{-i\Omega T S^z}$, $U_\Phi = e^{-i\Phi S^x}$, followed by computation of matrix exponentials.

(b) Bookwork (see Handout Problem 7.9).

(c)

$$U_\Omega U_\Phi = \begin{pmatrix} e^{-i\Omega T/2}\cos\Phi/2 & -ie^{-i\Omega T/2}\sin\Phi/2 \\ -ie^{i\Omega T/2}\sin\Phi/2 & e^{i\Omega T/2}\cos\Phi/2 \end{pmatrix}$$

from which we can read off $\cos\theta/2 = \cos\Omega T/2\cos\Phi/2$ and the direction vector

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1-\cos^2\Omega T/2\cos^2\Phi/2}}\begin{pmatrix} -\cos\Omega T/2\sin\Phi/2 \\ \sin\Omega T/2\sin\Phi/2 \\ -\sin\Omega T/2\cos\Phi/2 \end{pmatrix}$$

(d) There will be *no change*. The reason is that the rotation of a spin

$$\mathbf{S} \to e^{-i\boldsymbol{\theta}\cdot\mathbf{S}/\hbar}\mathbf{S}e^{i\boldsymbol{\theta}\cdot\mathbf{S}/\hbar}$$

depends only on the angular momentum algebra and not on the value of the spin. The spin-1/2 case just makes the computation of the matrix exponential easier.

2

(a) Show that

$$K(x,t|x',t') = \frac{\theta(t-t')}{\sqrt{4\pi D(t-t')}} \exp\left[-\frac{(x-x')^2}{4D(t-t')}\right]$$

is the fundamental solution of the diffusion equation in one dimension, meaning that it obeys

$$\left[\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right] K(x,t|x',t') = \delta(x-x')\delta(t-t')$$
$$K(x,t|x',t') = 0 \text{ for } t < t'.$$

[6]

(b) Show that $K(x,t|x',t')$ has the path integral representation

$$K(x_f,t_f|x_i,t_i) = \int_{\substack{x(t_\text{f})=x_\text{f}\\ x(t_\text{i})=x_\text{i}}} \mathcal{D}x(t)\, \exp\left(-\int_{t_\text{i}}^{t_\text{f}} \frac{\dot{x}^2}{4D}dt\right).$$ [8]

(c) $K(x_f,t_f|x_i,t_i)$ represents the probability that a particle diffuses from position x_i at time t_i to position x_f at time t_f. Explain why the probability to do so *without leaving the interval* $-L < x < L$ is given by

$$K_{|x|<L}(x_f,t_f|x_i,t_i) = \int_{\substack{x(t_\text{f})=x_\text{f}\\ x(t_\text{i})=x_\text{i}}} \mathcal{D}x(t)\, \exp\left(-\int_{t_\text{i}}^{t_\text{f}} \left[\frac{\dot{x}^2}{4D} + V(x)\right] dt\right),$$ [7]

where $V(x)$ is the infinite potential well potential

$$V(x) = \begin{cases} 0 & |x| < L \\ \infty & |x| \geq L. \end{cases}$$

(d) $K_{|x|<L}$ can be written in the form

$$K_{|x|<L}(x_f,t_f|x_i,t_i) = \theta(t_f - t_i)\sum_\alpha \varphi_\alpha(x_\text{f})\varphi^*_\alpha(x_\text{i})e^{-E_\alpha(t_\text{f}-t_\text{i})},$$

where $\varphi_\alpha(x)$ and E_α are the eigenfunctions and eigenvalues of the operator

$$H = -D\frac{\partial^2}{\partial x^2} + V(x).$$

Use this relationship to find an expression for $K_{|x|<L}(x,t|0,0)$. [8]

(e) Show that for large t the probability that a particle starting from $x = 0$ remains in $-L < x < L$ for the whole period is approximately

$$\frac{4}{\pi}\exp\left(-\frac{\pi^2 Dt}{4L^2}\right).$$ [4]

Solution 2. (a) Bookwork (Handout Problem 3.2)

(b) Key thing is reproducing property

$$K(x,t|x',t') = \int dx''\, K(x,t|x'',t'')K(x'',t''|x',t').$$

Break the interval up into many smaller intervals and approximate the exponent for each by $-\frac{\dot{x}^2}{4D}\Delta t$. In the limit this goes over to the path integral.

(c) Essentially the only thing that needs to be said is that the infinite well potential eliminates the contribution of any path that steps outside the interval.

(d) Eigenfunctions are

$$\begin{aligned} C_n &= \frac{1}{\sqrt{L}} \cos \frac{\pi n x}{2L}, \qquad n = 1,3,5,\ldots \\ S_n &= \frac{1}{\sqrt{L}} \sin \frac{\pi n x}{L}, \qquad n = 1,2,3,\ldots. \end{aligned}$$

However, the sines don't contribute to the propagator starting from $x = 0$. The 'energy' of the C_n is $E_n = \frac{D\pi^2 n^2}{4L^2}$, giving

$$K_{|x|<L}(x,t|0,0) = \sum_{n=0}^{\infty} \frac{1}{L} \cos \frac{\pi(2n+1)x}{2L} \exp\left(-\frac{D\pi^2(2n+1)^2 t}{4L^2}\right)$$

(e) Integrating over $[L,-L]$ and retaining only the first term gives the result.

3 (a) A plane wave incident in the $+z$ direction is scattered from a scattering potential of finite range located at the origin. Explain why the wavefunction has the asymptotic form

$$\psi_k(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \exp(ikz) + \frac{f_k(\theta,\phi)}{r}\exp(ikr)\,. \qquad (\star)$$ [6]

A model of scattering at energy

$$E_k = \frac{\hbar^2 k^2}{2m}$$

from a discrete state is given by the coupled equations

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_k(\mathbf{r}) + g d_k \delta(\mathbf{r}) &= E_k\psi_k(\mathbf{r}) \\ \varepsilon_0 d_k + g\psi_k(0) &= E_k d_k, \end{aligned} \qquad (\dagger)$$

where d_k is the amplitude of the wavefunction in the discrete state, and g represents the coupling between the discrete state and the continuum.

(b) By assuming $\psi_k(\mathbf{r})$ has the form ($\star$) for *all* $\mathbf{r}$, with f_k independent of angle, find the relationship between f_k and d_k from the first of equations ($\dagger$). [8]

Hint:

$$(\nabla^2 + k^2)\frac{e^{ikr}}{r} = -4\pi\delta(\mathbf{r})$$

(c) By taking $\psi_k(0)$ in the second of equations ($\dagger$) to be $\psi_k(\delta)$, for some small quantity δ, find a second relationship between f_k and d_k. [8]

(d) Combine your answers to parts (b) and (c) to show that

$$f_k = -\frac{\hbar\gamma}{\sqrt{2m}}\frac{1}{E_k - \tilde{\varepsilon}_0 + i\gamma\sqrt{E_k}}$$

where

$$\tilde{\varepsilon}_0 = \varepsilon_0 - \frac{m}{2\pi\hbar^2}\frac{g^2}{\delta},$$

and you should identify the quantity γ. [11]

Solution 3.

(a) Bookwork (Handout Page 35)

(b) Straightforward application of given identity

$$\frac{2\pi\hbar^2}{m}f_k + g d_k = 0$$

(c) By expanding the wavefunction

$$\psi_k(\delta) \sim \frac{f_k}{\delta} + (1 + ikf_k)$$

we obtain

$$\varepsilon_0 d_k + g\left(\frac{f_k}{\delta} + 1 + ikf\right) = E_k d$$

(d) Combining the two gives

$$(\tilde{\varepsilon}_0 - E_k)\,\frac{2\pi\hbar^2}{mg} f_k + g\,(1 + ikf_k) = 0$$

or

$$f_k = -\frac{g}{(E_k - \tilde{\varepsilon}_0)\,\frac{2\pi\hbar^2}{mg} + gik}$$

which yields the answer, with $\gamma = g^2 m^{3/2}/(\sqrt{2}\pi\hbar^3)$.

4

(a) Give the form of the density matrix describing a quantum system with Hamiltonian H in thermal equilibrium at temperature T. [6]

A pair of coupled oscillators are described by the Hamiltonian

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right) + kxy.$$

When $k = 0$ we may describe the system in terms of the usual ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p_x}{m\omega}\right)$$
$$b = \sqrt{\frac{m\omega}{2\hbar}}\left(y + i\frac{p_y}{m\omega}\right),$$

and their conjugates. For $k \neq 0$ we define new operators $\tilde{a}$, $\tilde{b}$ by

$$a = \tilde{a}\, \cosh\theta - \tilde{b}^\dagger\, \sinh\theta$$
$$b = \tilde{b}\, \cosh\theta - \tilde{a}^\dagger\, \sinh\theta.$$

(b) Find the value of θ that eliminates $\tilde{a}\tilde{b}$ and $\tilde{a}^\dagger\tilde{b}^\dagger$ terms from the Hamiltonian. [8]

(c) For this value of θ, the ground state for $k \neq 0$ satisfies $\tilde{a}\,|\text{g.s.}\rangle = \tilde{b}\,|\text{g.s.}\rangle = 0$. Show that

$$|\text{g.s.}\rangle = \mathcal{N}\exp(\lambda a^\dagger b^\dagger)\,|0\rangle_x\,|0\rangle_y\,,$$

satisfies these conditions, where $|0\rangle_x$ and $|0\rangle_y$ are the ground states of the x and y oscillators for $k = 0$, and $\mathcal{N}$ is a normalization factor that you do not need to find. Determine λ in terms of θ. [9]

(d) The *reduced* density matrix of the x oscillator is defined as

$$\rho_x = \sum_n \langle n|_y\left(|\text{g.s.}\rangle\,\langle\text{g.s.}|\right)|n\rangle_y\,,$$

where $|n\rangle_y = \frac{1}{\sqrt{n!}}(b^\dagger)^n\,|0\rangle_y$. Show that ρ_x describes an oscillator in thermal equilibrium, and find the temperature.

[10]

Solution 4.

(a) Bookwork (Handout Eqs. (5.36)-(5.37))

(b) Rewriting the Hamiltonian in terms of a, b, and then transforming to $\tilde{a}$, $\tilde{b}$, we find that the term in $\tilde{a}\tilde{b}$ is

$$-2\omega \sinh\theta \cosh\theta + \frac{k}{2mw}\left(\cosh^2\theta + \sinh^2\theta\right).$$

Setting this to zero gives

$$\tanh 2\theta = \frac{k}{2m\omega^2}.$$

(c) Apply $\tilde{a}$ to the given state:

$$\tilde{a}\,|\text{g.s.}\rangle = \left(a\cosh\theta + b^\dagger \sinh\theta\right)|\text{g.s.}\rangle = \left(\lambda b^\dagger \cosh\theta + b^\dagger \sinh\theta\right)|\text{g.s.}\rangle\,,$$

The last step follows from the same manipulation that proves that a coherent state $e^{\alpha a^\dagger}|0\rangle$ is an eigenstate of a. Equating to zero gives

$$\lambda = -\tanh\theta$$

(d) The ground state is

$$|\text{g.s.}\rangle = \mathcal{N}\sum_n \lambda^n\, |n\rangle_x\, |n\rangle_y\,.$$

Thus,

$$\rho_a = |\mathcal{N}|^2 \sum_n \lambda^{2n}\, |n\rangle_x\, \langle n|_x\,,$$

which means that $\rho_a = Z^{-1} e^{-\beta\hbar\omega a^\dagger a}$ with $\beta\hbar\omega = -\log\lambda^2$. Since $|\lambda| < 1$ this is always positive.

5 In a particular scattering experiment the S-matrix relating incoming $\alpha_{1,2}$ amplitudes to outgoing $\beta_{1,2}$ amplitudes is

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

(a) Explain why the S-matrix has to be a unitary matrix. [6]

$a_i^\dagger$, a_i $(i = 1, 2)$ create and destroy particles in the incoming states, and $b_i^\dagger$, b_i do the same for the outgoing states. The creation operators are related by the S-matrix

$$\begin{pmatrix}b_1^\dagger\\ b_2^\dagger\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_1^\dagger\\ a_2^\dagger\end{pmatrix}.$$

In the following parts, give all possible occupancies of outgoing states 1 and 2, *and their probabilities*. The statistics of the particles are given in square brackets. Note that the states are not necessarily normalized.

(b) The initial state is $a_1^\dagger a_2^\dagger |\mathrm{VAC}\rangle$ [Fermions]. [6]

(c) The initial state is $a_1^\dagger a_2^\dagger |\mathrm{VAC}\rangle$ [Bosons]. [6]

(d) The initial state is $(a_1^\dagger)^N (a_2^\dagger)^N |\mathrm{VAC}\rangle$ [Bosons]. [7]

(e) The initial state is $\exp(\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger) |\mathrm{VAC}\rangle$ [Bosons]. [8]

Solution 5.

(a) Bookwork. Some discussion of flux conservation (see Handout Page 31).

(b) Outgoing state is $\frac{1}{2}(b_1^\dagger + b_2^\dagger)(b_1^\dagger - b_2^\dagger) |\mathrm{VAC}\rangle = -b_1^\dagger b_2^\dagger |\mathrm{VAC}\rangle$. Only possible occupancies are $N_1 = N_2 = 1$.

(c) Outgoing state is $\frac{1}{2}(b_1^\dagger + b_2^\dagger)(b_1^\dagger - b_2^\dagger) |\mathrm{VAC}\rangle = \frac{1}{2}\left((b_1^\dagger)^2 - (b_2^\dagger)^2\right) |\mathrm{VAC}\rangle$. Either $N_1 = 2$ $N_2 = 0$ or $N_1 = 0$, $N_2 = 1$, both with probability 1/2.

(d) Outgoing state (after normalization) is $\frac{1}{2^N\sqrt{2N!}}(b_1^\dagger + b_2^\dagger)^N(b_1^\dagger - b_2^\dagger)^N |\mathrm{VAC}\rangle = \frac{1}{2^N\sqrt{N!}}\left((b_1^\dagger)^2 - (b_2^\dagger)^2\right)^N |\mathrm{VAC}\rangle$. Probability distribution is

$$P(N_1, N_2) = \begin{cases} 0 & N_{1,2} = \text{ odd} \\ \frac{N_1! N_2!}{2^{2N}(2N)!}\left(\frac{N!}{(N_1/2)!(N_2/2)!}\right)^2 & N_{1,2} = \text{ even}\end{cases}$$

(e) The final state is

$$\exp\left(\frac{1}{\sqrt{2}}\left[\alpha_1+\alpha_2\right]b_1^\dagger+\frac{1}{\sqrt{2}}\left[\alpha_1-\alpha_2\right]b_2^\dagger\right)|\mathrm{VAC}\rangle$$

(unnormalized). Since this is a product of coherent states, the distribution is independent Poissonian for each state with rates $\frac{1}{2}|\alpha_1 \pm \alpha_2|^2$.

Proof for a single coherent state: $e^{-|\alpha|^2/2}e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$ which gives probabilities $P_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$.

6

(a) State the properties satisfied by the matrices S that form the Lie group $SL(2,\mathbb{C})$. [3]

(b) The hermitian matrix

$$\mathsf{X} = \begin{pmatrix} ct+z & x+iy \\ x-iy & ct-z \end{pmatrix}$$

is associated with the four vector $x^\mu = (ct, x, y, z)$. Explain why the transformation

$$\mathsf{X} \to \mathsf{X}' = \mathsf{S}\mathsf{X}\mathsf{S}^\dagger,$$

corresponds to a Lorentz transformation of the four vector. [*Hint: consider the determinant.*] [7]

(c) Show that *light-like* four vectors may be parameterized in terms of a two component spinor by

$$\mathsf{X}_{AB} = \chi_A \chi_B^*, \qquad A, B = 1, 2,$$

and that

$$\chi = \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix},$$

corresponds to a light-like four vector with spatial part pointing in a direction specified by the polar coordinates (θ, ϕ) in the usual way. [7]

(d) Find how θ and ϕ transform for $\mathsf{S} = \exp(\kappa\sigma_z/2)$. [7]

(e) Show that your answer to (d) is consistent with the transformation of the four vector found in part (c) by the Lorentz transformation $\Lambda = \exp(\kappa \mathsf{K}_z)$, where

$$\mathsf{K}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

[9]

Solution 6.

(a) 2×2 complex matrices of unit determinant (see Handout Page 106).

(b) $\det \mathsf{X} = (ct)^2 - \mathbf{x}^2$, which is preserved by the given transformations, which also preserves hermiticity. Linear transform that preserves invariant interval = Lorentz transformation.

(c) X of this form have vanishing determinant. Note that $\chi_A \xi_B$ has vanishing determinant but $\xi = \chi^*$ is required for X to be hermitian. Explicit computation of

4-vector gives

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

`Here there was a sign error` $\phi \to -\phi$ `in the question!`

(d) Under $\chi \to \chi' = \mathsf{S}\chi$ ϕ is unchanged and

$$\tan\theta'/2 = e^{-\kappa}\tan\theta/2$$

(e) Finding the matrix exponential gives

$$\Lambda = \begin{pmatrix} \cosh\kappa & 0 & 0 & \sinh\kappa \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\kappa & 0 & 0 & \cosh\kappa \end{pmatrix}.$$

Transforming the light-like four-vector from the previous part gives $x' = x$, $y = y'$, and

$$\begin{aligned} t' &= \cosh\kappa + z\sinh\kappa \\ z' &= z\cosh\kappa + \sinh\kappa\text{'} \end{aligned}$$

so that

$$\tan\theta' = \frac{\sin\theta}{\cosh\kappa\cos\theta + \sinh\kappa}.$$

Applying some trig we write this as

$$\frac{2\tan\theta'/2}{1-\tan^2\theta'/2} = \frac{2\sin\theta/2\cos\theta/2}{\cosh\kappa(2\cos^2\theta/2 - 1) + \sinh\kappa} = \frac{2e^{-\kappa}\tan\theta/2}{1 - e^{-2\kappa}\tan^2\theta/2},$$

which agrees with the result of part (d).

END OF PAPER