UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

22 April 2015, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***three*** *questions only. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 A spin-1/2 evolves according to the time-dependent Hamiltonian

$$H(t) = \hbar\Omega S^z + \hbar\Phi S^x \sum_{n=0}^{\infty} \delta(t - nT),$$

where $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$.

(a) Show that the evolution operator from just before the first 'pulse' to just before the $(N+1)^{\text{th}}$, is given by

$$U(NT - \epsilon, -\epsilon) = (U_\Omega U_\Phi)^N,$$

where

$$U_\Omega = \begin{pmatrix} e^{-i\Omega T/2} & 0 \\ 0 & e^{i\Omega T/2} \end{pmatrix}, \qquad U_\Phi = \begin{pmatrix} \cos\Phi/2 & -i\sin\Phi/2 \\ -i\sin\Phi/2 & \cos\Phi/2 \end{pmatrix}.$$ [7]

(b) Show that, if $\boldsymbol{\theta} = \theta\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector

$$\exp\left(-i\boldsymbol{\theta}\cdot\mathbf{S}\right) = \mathbb{1}\cos\left(\theta/2\right) - i\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}\sin\left(\theta/2\right).$$ [7]

(c) Apply the result of (b) to the evolution operator in (a) to find the angle through which the spin rotates, and its axis of rotation. [12]

(d) How would the answer to part (c) change if we were to consider a spin S instead? Explain your reasoning. [7]

2

(a) Show that

$$K(x,t|x',t') = \frac{\theta(t-t')}{\sqrt{4\pi D(t-t')}} \exp\left[-\frac{(x-x')^2}{4D(t-t')}\right]$$

is the fundamental solution of the diffusion equation in one dimension, meaning that it obeys

$$\left[\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right] K(x,t|x',t') = \delta(x-x')\delta(t-t')$$
$$K(x,t|x',t') = 0 \text{ for } t < t'.$$

[6]

(b) Show that $K(x,t|x',t')$ has the path integral representation

$$K(x_f,t_f|x_i,t_i) = \int_{\substack{x(t_\mathrm{f})=x_\mathrm{f} \\ x(t_\mathrm{i})=x_\mathrm{i}}} \mathcal{D}x(t) \exp\left(-\int_{t_\mathrm{i}}^{t_\mathrm{f}} \frac{\dot{x}^2}{4D} dt\right).$$ [8]

(c) $K(x_f,t_f|x_i,t_i)$ represents the probability that a particle diffuses from position x_i at time t_i to position x_f at time t_f. Explain why the probability to do so *without leaving the interval* $-L < x < L$ is given by

$$K_{|x|<L}(x_f,t_f|x_i,t_i) = \int_{\substack{x(t_\mathrm{f})=x_\mathrm{f} \\ x(t_\mathrm{i})=x_\mathrm{i}}} \mathcal{D}x(t) \exp\left(-\int_{t_\mathrm{i}}^{t_f} \left[\frac{\dot{x}^2}{4D} + V(x)\right] dt\right),$$ [7]

where $V(x)$ is the infinite potential well potential

$$V(x) = \begin{cases} 0 & |x| < L \\ \infty & |x| \geq L. \end{cases}$$

(d) $K_{|x|<L}$ can be written in the form

$$K_{|x|<L}(x_f,t_f|x_i,t_i) = \theta(t_f - t_i)\sum_\alpha \varphi_\alpha(x_\mathrm{f})\varphi^*_\alpha(x_\mathrm{i})e^{-E_\alpha(t_\mathrm{f}-t_\mathrm{i})},$$

where $\varphi_\alpha(x)$ and E_α are the eigenfunctions and eigenvalues of the operator

$$H = -D\frac{\partial^2}{\partial x^2} + V(x).$$

Use this relationship to find an expression for $K_{|x|<L}(x,t|0,0)$. [8]

(e) Show that for large t the probability that a particle starting from $x = 0$ remains in $-L < x < L$ for the whole period is approximately

$$\frac{4}{\pi}\exp\left(-\frac{\pi^2 Dt}{4L^2}\right).$$ [4]

3 (a) A plane wave incident in the $+z$ direction is scattered from a scattering potential of finite range located at the origin. Explain why the wavefunction has the asymptotic form

$$\psi_k(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \exp(ikz) + \frac{f_k(\theta,\phi)}{r}\exp(ikr)\,. \qquad (\star)$$ [6]

A model of scattering at energy

$$E_k = \frac{\hbar^2 k^2}{2m}$$

from a discrete state is given by the coupled equations

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_k(\mathbf{r}) + g d_k \delta(\mathbf{r}) &= E_k\psi_k(\mathbf{r}) \\ \varepsilon_0 d_k + g\psi_k(0) &= E_k d_k, \end{aligned} \qquad (\dagger)$$

where d_k is the amplitude of the wavefunction in the discrete state, and g represents the coupling between the discrete state and the continuum.

(b) By assuming $\psi_k(\mathbf{r})$ has the form $(\star)$ for *all* $\mathbf{r}$, with f_k independent of angle, find the relationship between f_k and d_k from the first of equations $(\dagger)$. [8]

Hint:

$$(\nabla^2 + k^2)\frac{e^{ikr}}{r} = -4\pi\delta(\mathbf{r})$$

(c) By taking $\psi_k(0)$ in the second of equations $(\dagger)$ to be $\psi_k(\delta)$, for some small quantity δ, find a second relationship between f_k and d_k. [8]

(d) Combine your answers to parts (b) and (c) to show that

$$f_k = -\frac{\hbar\gamma}{\sqrt{2m}}\frac{1}{E_k - \tilde{\varepsilon}_0 + i\gamma\sqrt{E_k}}$$

where

$$\tilde{\varepsilon}_0 = \varepsilon_0 - \frac{m}{2\pi\hbar^2}\frac{g^2}{\delta},$$

and you should identify the quantity γ. [11]

V7.1

A

4

(a) Give the form of the density matrix describing a quantum system with Hamiltonian H in thermal equilibrium at temperature T. [6]

A pair of coupled oscillators are described by the Hamiltonian

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right) + kxy.$$

When $k = 0$ we may describe the system in terms of the usual ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p_x}{m\omega}\right)$$
$$b = \sqrt{\frac{m\omega}{2\hbar}}\left(y + i\frac{p_y}{m\omega}\right),$$

and their conjugates. For $k \neq 0$ we define new operators $\tilde{a}$, $\tilde{b}$ by

$$a = \tilde{a}\,\cosh\theta - \tilde{b}^\dagger\,\sinh\theta$$
$$b = \tilde{b}\,\cosh\theta - \tilde{a}^\dagger\,\sinh\theta.$$

(b) Find the value of θ that eliminates $\tilde{a}\tilde{b}$ and $\tilde{a}^\dagger\tilde{b}^\dagger$ terms from the Hamiltonian. [8]

(c) For this value of θ, the ground state for $k \neq 0$ satisfies $\tilde{a}\,|\text{g.s.}\rangle = \tilde{b}\,|\text{g.s.}\rangle = 0$. Show that

$$|\text{g.s.}\rangle = \mathcal{N}\exp(\lambda a^\dagger b^\dagger)\,|0\rangle_x\,|0\rangle_y\,,$$

satisfies these conditions, where $|0\rangle_x$ and $|0\rangle_y$ are the ground states of the x and y oscillators for $k = 0$, and $\mathcal{N}$ is a normalization factor that you do not need to find. Determine λ in terms of θ. [9]

(d) The *reduced* density matrix of the x oscillator is defined as

$$\rho_x = \sum_n \langle n|_y \left(|\text{g.s.}\rangle\,\langle\text{g.s.}|\right)|n\rangle_y\,,$$

where $|n\rangle_y = \frac{1}{\sqrt{n!}}(b^\dagger)^n\,|0\rangle_y$. Show that ρ_x describes an oscillator in thermal equilibrium, and find the temperature.

[10]

5 In a particular scattering experiment the S-matrix relating incoming $\alpha_{1,2}$ amplitudes to outgoing $\beta_{1,2}$ amplitudes is

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

(a) Explain why the S-matrix has to be a unitary matrix. [6]

$a_i^\dagger$, a_i $(i = 1, 2)$ create and destroy particles in the incoming states, and $b_i^\dagger$, b_i do the same for the outgoing states. The creation operators are related by the S-matrix

$$\begin{pmatrix}b_1^\dagger\\ b_2^\dagger\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_1^\dagger\\ a_2^\dagger\end{pmatrix}.$$

In the following parts, give all possible occupancies of outgoing states 1 and 2, *and their probabilities.* The statistics of the particles are given in square brackets. Note that the states are not necessarily normalized.

(b) The initial state is $a_1^\dagger a_2^\dagger \left|\mathrm{VAC}\right\rangle$ [Fermions]. [6]

(c) The initial state is $a_1^\dagger a_2^\dagger \left|\mathrm{VAC}\right\rangle$ [Bosons]. [6]

(d) The initial state is $(a_1^\dagger)^N (a_2^\dagger)^N \left|\mathrm{VAC}\right\rangle$ [Bosons]. [7]

(e) The initial state is $\exp(\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger) \left|\mathrm{VAC}\right\rangle$ [Bosons]. [8]

6

(a) State the properties satisfied by the matrices S that form the Lie group $SL(2,\mathbb{C})$. [3]

(b) The hermitian matrix

$$\mathsf{X} = \begin{pmatrix} ct+z & x+iy \\ x-iy & ct-z \end{pmatrix}$$

is associated with the four vector $x^\mu = (ct, x, y, z)$. Explain why the transformation

$$\mathsf{X} \to \mathsf{X}' = \mathsf{S}\mathsf{X}\mathsf{S}^\dagger,$$

corresponds to a Lorentz transformation of the four vector. [*Hint: consider the determinant.*] [7]

(c) Show that *light-like* four vectors may be parameterized in terms of a two component spinor by

$$\mathsf{X}_{AB} = \chi_A \chi_B^*, \qquad A, B = 1, 2,$$

and that

$$\chi = \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix},$$

corresponds to a light-like four vector with spatial part pointing in a direction specified by the polar coordinates (θ, ϕ) in the usual way. [7]

(d) Find how θ and ϕ transform for $\mathsf{S} = \exp(\kappa\sigma_z/2)$. [7]

(e) Show that your answer to (d) is consistent with the transformation of the four vector found in part (c) by the Lorentz transformation $\Lambda = \exp(\kappa \mathsf{K}_z)$, where

$$\mathsf{K}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

[9]

END OF PAPER