UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

20 April 2016, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***all four*** *questions. Each question consists of 5 parts, worth 5 marks each. The paper contains seven sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 The time-dependent Hamiltonian

$$H = -\frac{\hbar^2}{2I}\frac{d^2}{d\theta^2} - \hbar\eta\cos(\theta)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

describes the free motion of a particle of mass M on a ring of radius R, where $I = MR^2$ is the moment of inertia, interrupted by 'kicks' at times $t = 0, \pm T, \pm 2T, \ldots$.

(a) Show that the evolution operator from a time 0+ (that is, infinitesimally after time $t = 0$) to a time $NT+$ may be written as

$$U(0+ \to NT+) = (U_\eta U_T)^N\,, \tag{1}$$

where

$$U_T = \exp\left(\frac{i\hbar T}{2I}\frac{d^2}{d\theta^2}\right), \qquad U_\eta = \exp\left(i\eta\cos\theta\right). \tag{2}$$

(b) By interpreting the function

$$\exp\left(ikr\cos\theta\right) \tag{3}$$

as a plane wave in 2D polar coordinates, show that it has the expansion

$$\exp\left(ikr\cos\theta\right) = \sum_{m=-\infty}^{\infty} a_m J_m(kr)e^{im\theta}, \tag{4}$$

where $J_m(\rho)$ is the Bessel function satisfying

$$\rho^2\frac{d^2}{d\rho^2}J_m + \rho\frac{d}{d\rho}J_m + \left(\rho^2 - m^2\right)J_m = 0, \tag{5}$$

and a_m are some coefficients to be determined below.

(c) Given the behaviour

$$J_m(\rho) \to \frac{\rho^m}{2^m m!} \text{ as } \rho \to 0, \tag{6}$$

find the coefficients a_m in the previous part.

(d) Find the expression for the action of $U_\eta U_T$ on the Fourier components c_m of the wavefunction

$$\psi(\theta, t) = \sum_{m=-\infty}^{\infty} c_m(t)e^{im\theta} \tag{7}$$

(e) Show that if

$$\frac{\hbar T}{2I} = 2\pi, \tag{8}$$

then U_T has no effect on the time evolution. In this case, find the expectation value of the kinetic energy of the particle after N steps, assuming it starts in the $m = 0$ ground state.

Solution 1. (a) This is just a question of evaluating $U = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{0+}^{NT+} H(t)dt\right)$ carefully. (b) We look for separable solutions of the Helmholtz equation

$$\left[\nabla^2 + k^2\right] \Phi(r,\theta) = 0, \tag{9}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Separable solutions are then shown to have the form $\Phi(r,\theta) = J_m(kr)e^{im\theta}$. The expansion of the plane wave then follows.

(c) This follows closely the 3D case. The key point is that a term $(kr)^m e^{im\theta}$ arises only from the m^{th} order. Using the given asymptote and comparing coefficients then gives $a_m = i^m$

(d) Applying $U_\eta U_T$ to the given expansion

$$\begin{aligned} U_\eta U_T \psi(\theta,t) &= U_\eta \sum_m c_m \exp\left(im\theta - \frac{i\hbar T m^2}{2I}\right) \\ &= \sum_{m'=-\infty}^{\infty} i^{m'} J_{m'}(kr) e^{im'\theta} \sum_m c_m \exp\left(im\theta - \frac{i\hbar T m^2}{2I}\right). \end{aligned} \tag{10}$$

This shows that

$$c_m \to \sum_{m'} i^{m-m'} J_{m-m'} c_{m'} e^{-i\hbar T m'^2/2I}. \tag{11}$$

(e) When the given condition is satisfied $U_T = \mathbb{1}$, so that $U = U_\eta^N$. The expectation value of the energy is

$$\langle E\rangle = \frac{\hbar^2}{2I}\int_0^{2\pi} |\partial_\theta e^{i\eta N\cos\theta}|^2 \frac{d\theta}{2\pi} = \frac{(\hbar\eta N)^2}{2I}\int_0^{2\pi} \sin^2\theta \frac{d\theta}{2\pi} = \frac{(\hbar\eta N)^2}{4I}, \tag{12}$$

showing that the energy increases quadratically with the number of steps. Parenthetically, moving away from resonance causes diffusion and eventually localization of the eigenstates.

2 A system consists of two spin-1/2 subsystems, labelled A and B, with states denoted by $|s\rangle_{A,B}$, with $s = \uparrow, \downarrow$ corresponding to eigenvalues of the Pauli matrix $\sigma^z_{A,B}$ equal to ± 1.

A general state of the composite system may be written

$$|\chi\rangle = a_1 |\uparrow\rangle_A |\uparrow\rangle_B + a_2 |\uparrow\rangle_A |\downarrow\rangle_B + a_3 |\downarrow\rangle_A |\uparrow\rangle_B + a_4 |\downarrow\rangle_A |\downarrow\rangle_B \tag{13}$$

(a) Show that the reduced density matrix of the A subsystem $\rho_A \equiv \mathrm{tr}_B |\chi\rangle \langle\chi|$ is

$$\rho_A = \begin{pmatrix} |a_1|^2 + |a_2|^2 & a_1 a_3^* + a_2 a_4^* \\ a_1^* a_3 + a_2^* a_4 & |a_3|^2 + |a_4|^2 \end{pmatrix}. \tag{14}$$

(b) Consider the alternative basis of states

$$\begin{aligned} |1\rangle &= \frac{i}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B \right) \\ |2\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B \right) \\ |3\rangle &= \frac{i}{\sqrt{2}} \left(|\uparrow\rangle_A |\uparrow\rangle_B - |\downarrow\rangle_A |\downarrow\rangle_B \right) \\ |4\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right). \end{aligned} \tag{15}$$

Writing the general state now as

$$|\chi\rangle = b_1 |1\rangle + b_2 |2\rangle + b_3 |3\rangle + b_4 |4\rangle ,$$

show that the determinant of ρ_A is

$$\det \rho_A = \frac{1}{4} \left| \sum_{i=1}^{4} b_i^2 \right|^2. \tag{16}$$

Find the eigenvalues of the density matrix and the entanglement entropy $S_A = -\,\mathrm{tr}\left[\rho_A \log \rho_A\right]$.

(c) What are the conditions satisfied by a matrix belonging to the Lie group $SO(4)$? Explain why the matrices

$$\begin{aligned} &\mathsf{L}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{L}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathsf{L}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\mathsf{K}_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{K}_y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{K}_z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{17}$$

form a basis for the Lie algebra $so(4)$.

(d) The algebra of these generators is

$$\begin{aligned}[\mathsf{L}_a, \mathsf{L}_b] &= \varepsilon_{abc}\mathsf{L}_c \\ [\mathsf{L}_a, \mathsf{K}_b] &= \varepsilon_{abc}\mathsf{K}_c\,, \qquad \text{(summation implied)} \\ [\mathsf{K}_a, \mathsf{K}_b] &= \varepsilon_{abc}\mathsf{L}_c.\end{aligned} \tag{18}$$

Find the commutation relations satisfied by $L_a^{\pm} \equiv (L_a \pm K_a)/2$ and interpret this result.

(e) By considering the action of $\sigma_A^{x,y,z}$ on the states in part (b), explain why the entanglement entropy is independent of unitary transformations of the A spin only.

Solution 2. (a) We have

$$\begin{aligned}\rho_A = \mathrm{tr}_B\,&[(a_1\,|{\uparrow}\rangle_A\,|{\uparrow}\rangle_B + a_2\,|{\uparrow}\rangle_A\,|{\downarrow}\rangle_B + a_3\,|{\downarrow}\rangle_A\,|{\uparrow}\rangle_B + a_4\,|{\downarrow}\rangle_A\,|{\downarrow}\rangle_B) \\ &(a_1^*\,\langle{\uparrow}|_A\,\langle{\uparrow}|_B + a_2^*\,\langle{\uparrow}|_A\,\langle{\downarrow}|_B + a_3^*\,\langle{\downarrow}|_A\,\langle{\uparrow}|_B + a_4^*\,\langle{\downarrow}|_A\,\langle{\downarrow}|_B)]\,.\end{aligned} \tag{19}$$

Taking the trace over the B states yields the answer.

(b) The determinant of ρ_A is $|a_1a_4 - a_2a_3|^2$. Finding a_i in terms of b_i yields the given answer. $\det\rho_A = \lambda_+\lambda_-$, the product of the eigenvalues, and since $\lambda_+ + \lambda_- = 1$, we have

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\det\rho_A}\right).$$

The entanglement entropy is just $S_A = -\lambda_+\log\lambda_+ - \lambda_-\log\lambda_-$. There's no need to write these expressions out in more detail.

(c) Matrix is 4×4, orthogonal with determinant $+1$. If $M = \exp(\Lambda)$ satisfies this condition, then Λ is antisymmetric, and the given matrices form a basis for the antisymmetric matrices.

(d) This is just taking linear combinations to give

$$\begin{aligned}\left[\mathsf{L}_a^{\pm}, \mathsf{L}_b^{\pm}\right] &= \varepsilon_{abc}\mathsf{L}_c^{\pm} \\ \left[\mathsf{L}_a^{\pm}, \mathsf{L}_b^{\mp}\right] &= 0.\end{aligned} \tag{20}$$

Interpretation: this is two copies of the $su(2) \sim so(3)$ algebra, showing that $SO(4) \sim SU(2)\times SU(2)$

(e) The action of the three Pauli matrices is

$$\begin{aligned}
\sigma_A^x \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix} &= \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix} \\
\sigma_A^y \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix} \\
\sigma_A^z \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{pmatrix}
\end{aligned} \tag{21}$$

This coincides with $iL^-_{x,y,z}$, showing that the transformations generated are elements of $SO(4)$. The crucial point is that such transformations leave the quadratic form $\sum_{j=1}^4 a_j^2$ invariant. Hence the determinant of ρ_A, and the eigenvalues, are unchanged.

3 A Hamiltonian describing bosons with spin that can move between two sites is

$$H = \overbrace{t \sum_{s=\uparrow,\downarrow} \left(a_s^\dagger b_s + b_s^\dagger a_s\right)}^{H_t} + \mathcal{E}(\Delta) \tag{22}$$

where $a_s^\dagger$, a_s, $b_s^\dagger$, b_s are creation and annihilation operators for the A and B sites, $N_a = \sum_{s=\uparrow,\downarrow} a_s^\dagger a_s$, $N_b = \sum_{s=\uparrow,\downarrow} b_s^\dagger b_s$ and $\mathcal{E}(N_a - N_b)$ is some function of the number difference $\Delta \equiv N_a - N_b$.

(a) Show that

$$\begin{aligned} \left[a_s^\dagger b_s, \mathcal{E}(\Delta)\right] &= \left(\mathcal{E}(\Delta - 2) - \mathcal{E}(\Delta)\right) a_s^\dagger b_s \\ \left[b_s^\dagger a_s, \mathcal{E}(\Delta)\right] &= \left(\mathcal{E}(\Delta + 2) - \mathcal{E}(\Delta)\right) b_s^\dagger a_s. \end{aligned} \tag{23}$$

[Hint: It might be easier to explain why $a_s^\dagger b_s \mathcal{E}(\Delta) = \mathcal{E}(\Delta - 2) a_s^\dagger b_s$*]*

(b) A unitary transformation $H \to H' \equiv e^S H e^{-S}$ is performed to remove H_t from the Hamiltonian at lowest order. Show that S must be chosen so that

$$[S, \mathcal{E}(\Delta)] = -H_t \tag{24}$$

(c) By taking S to have the form

$$S = f(\Delta) \sum_{s=\uparrow,\downarrow} a_s^\dagger b_s - \text{h.c.}, \tag{25}$$

(h.c. denotes the hermitian conjugate), find the function $f(\Delta)$.

(d) Show that the transformed Hamiltonian H' contains the term

$$H^{(2)} = \frac{1}{2} [S, H_t] \tag{26}$$

of order t^2. Evaluate the part of $H^{(2)}$ that does not change the occupancy of the two sites, leaving your answer expressed in terms of $f(\Delta)$.

(e) Discuss how the form of $H^{(2)}$ depends on whether $\mathcal{E}(\Delta)$ is linear (no interactions between particles) or not.

Solution 3. (a) This follows from the fundamental relations

$$[a^\dagger, N_a] = -a^\dagger, \qquad [a, N_a] = a \tag{27}$$

(b) We have

$$H' = e^S H e^{-S} = H + [S, H] + \frac{1}{2}[S, [S, H]] + \dots. \tag{28}$$

The given equation eliminates H_t from the Hamiltonian.

(c) Plugging in the given form, we need to find

$$[f(\Delta)a_s^\dagger b_s, \mathcal{E}(\Delta)] = f(\Delta)\left(\mathcal{E}(\Delta-2) - \mathcal{E}(\Delta)\right)a_s^\dagger b_s. \tag{29}$$

from which we immediately get

$$f(\Delta) = -\frac{t}{\mathcal{E}(\Delta-2) - \mathcal{E}(\Delta)}.$$

(d) At second order we have the terms

$$[S, H_t] + \frac{1}{2}[S, [S, \mathcal{E}(\Delta)]] = \frac{1}{2}[S, H_t],$$

where we use the relation defining S. The part which doesn't change the occupancy comes from the commutator

$$[f(\Delta)a_s^\dagger b_s, b_{s'}^\dagger a_{s'}] = \delta_{ss'} f(\Delta)\Delta + \left(f(\Delta) - f(\Delta+2)\right) b_{s'}^\dagger a_{s'} a_s^\dagger b_s. \tag{30}$$

The second order Hamiltonian is then

$$H^{(2)} = t^2 \left[f(\Delta)\Delta + \left(f(\Delta) - f(\Delta+2)\right) \sum_{s,s'} b_{s'}^\dagger a_{s'} a_s^\dagger b_s \right] \tag{31}$$

(e) The first term in $H^{(2)}$ is always present. The second term requires that $f(\Delta)$ is Δ dependent, which means that it only appears if $\mathcal{E}(\Delta)$ is nonlinear.

4 The Klein–Gordon equation is

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\Psi(\mathbf{r},t) = 0. \tag{32}$$

(a) Explain some of the difficulties in interpreting $\Psi(\mathbf{r},t)$ as a wavefunction, in contrast to solutions of Schrödinger's equation.

(b) A vector potential is introduced by the replacement $\nabla \to \nabla - ie\mathbf{A}/\hbar$. Show that the equation obeyed by the amplitude $\psi_{\mathbf{k}}(t)$ of a plane wave $\Psi(\mathbf{r},t) = \psi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}}$ in a spatially constant vector potential $\mathbf{A}(t)$ is

$$\ddot{\psi}_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}-e\mathbf{A}(t)/\hbar}\psi_{\mathbf{k}}(t) = 0, \tag{33}$$

where you should state the form of $\omega_{\mathbf{k}}$.

(c) Since $\mathbf{E} = -\dot{\mathbf{A}}$, an electric field pulse at time $t = 0$ is described by a vector potential

$$\mathbf{A}(t) = \begin{cases} 0 & t < 0 \\ \mathbf{A}_0 & t \geq 0 \end{cases} \tag{34}$$

Find the form of $\psi_{\mathbf{k}}(t)$ at $t > 0$ if at $t < 0$

$$\psi_{\mathbf{k}}(t) = \sqrt{\frac{1}{2\omega_{\mathbf{k}}}}a_{\mathbf{k}}\exp\left(-i\omega_{\mathbf{k}}t\right), \tag{35}$$

[*Hint: think of this as a 1D scattering problem*]

(d) By interpreting the positive and negative frequency parts of the $t > 0$ solution in terms of $a_{\mathbf{k}}$ and $b^\dagger_{-\mathbf{k}}$, show that the vacuum condition

$$a_{\mathbf{k}}\left|\mathrm{VAC}\right\rangle = 0, \qquad t < 0 \tag{36}$$

becomes

$$\left(u_{\mathbf{k}}a_{\mathbf{k}} + v_{\mathbf{k}}b^\dagger_{-\mathbf{k}}\right)\left|\mathrm{VAC}\right\rangle = 0, \qquad t > 0, \tag{37}$$

where you should find $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$.

(e) Show that $\left|\mathrm{VAC}\right\rangle$ differs from the state with no $t > 0$ particles or antiparticles by a factor

$$\mathcal{N}\prod_{\mathbf{k}}\exp\left(-(v_{\mathbf{k}}/u_{\mathbf{k}})a^\dagger_{\mathbf{k}}b^\dagger_{-\mathbf{k}}\right) \tag{38}$$

with some normalization $\mathcal{N}$, and find the probability distribution of $(\mathbf{k}, -\mathbf{k})$ particle-antiparticle pairs in terms of $v_{\mathbf{k}}/u_{\mathbf{k}}$.

Solution 4. (a) Negative frequency states; second order means initial Ψ not sufficient; physical meaning of $\Pi = \partial_t \Psi$; probability nonconservation, etc.

(b) Straightforward substitution. $\omega_{\mathbf{k}} = \sqrt{c^2\mathbf{k}^2 + (mc^2/\hbar)^2}$

(c) We have to solve

$$\begin{aligned} \ddot{\psi}_{\mathbf{k}} + \omega_-^2 \psi_{\mathbf{k}} = 0, \qquad & t < 0 \\ \ddot{\psi}_{\mathbf{k}} + \omega_+^2 \psi_{\mathbf{k}} = 0, \qquad & t > 0 \end{aligned} \tag{39}$$

where $\omega_+ = \omega_{\mathbf{k}}$, $\omega_- = \omega_{\mathbf{k}-e\mathbf{A}_0/\hbar}$. This corresponds to scattering off a potential step. From the continuity of the solution and its first derivative, the solution is

$$\psi_{\mathbf{k}}(t) = \begin{cases} \sqrt{\frac{1}{2\omega_-}} a_{\mathbf{k}} \exp\left(-i\omega_- t\right) & t < 0 \\ \sqrt{\frac{1}{8\omega_-}} a_{\mathbf{k}} \left[\left(1 - \frac{\omega_-}{\omega_+}\right) \exp\left(i\omega_+ t\right) + \left(1 + \frac{\omega_-}{\omega_+}\right) \exp\left(-i\omega_+ t\right)\right] & t \geq 0. \end{cases} \tag{40}$$

(d) From the solution we obtain

$$v = \frac{1}{2}\left(\sqrt{\frac{\omega_+}{\omega_-}} - \sqrt{\frac{\omega_-}{\omega_+}}\right), \qquad u = \frac{1}{2}\left(\sqrt{\frac{\omega_+}{\omega_-}} + \sqrt{\frac{\omega_-}{\omega_+}}\right) \tag{41}$$

(e) The basic observation is that since

$$a \exp(\alpha a^\dagger) \left|\mathrm{VAC}\right\rangle = \alpha \exp(\alpha a^\dagger) \left|\mathrm{VAC}\right\rangle, \tag{42}$$

(coherent states), the same is true when α is replaced by any operator commuting with a. For the probability distribution, recall that the normalized occuption number states are

$$\frac{1}{\sqrt{n!}} (a^\dagger)^n \left|\mathrm{VAC}\right\rangle. \tag{43}$$

Expanding the exponential shows us the probability of getting n pairs is then $\propto |u/v|^{2n}$. After normalizing this geometric distribution we get

$$P_n = (1 - |u/v|^2)|u/v|^{2n}.$$

END OF PAPER