UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

20 April 2016, 10.30 to 12.30

THEORETICAL PHYSICS 2

*Answer **all four** questions. Each question consists of 5 parts, worth 5 marks each. The paper contains six sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

1 The time-dependent Hamiltonian

$$H = -\frac{\hbar^2}{2I}\frac{d^2}{d\theta^2} - \hbar\eta\cos(\theta)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

describes the free motion of a particle of mass M on a ring of radius R, where $I = MR^2$ is the moment of inertia, interrupted by 'kicks' at times $t = 0, \pm T, \pm 2T, \ldots$.

(a) Show that the evolution operator from a time 0+ (that is, infinitesimally after time $t = 0$) to a time $NT+$ may be written as

$$U(0+ \to NT+) = \left(U_\eta U_T\right)^N,$$

where

$$U_T = \exp\left(\frac{i\hbar T}{2I}\frac{d^2}{d\theta^2}\right), \qquad U_\eta = \exp\left(i\eta\cos\theta\right).$$

(b) By interpreting the function

$$\exp\left(ikr\cos\theta\right)$$

as a plane wave in 2D polar coordinates, show that it has the expansion

$$\exp\left(ikr\cos\theta\right) = \sum_{m=-\infty}^{\infty} a_m J_m(kr)e^{im\theta},$$

where $J_m(\rho)$ is the Bessel function satisfying

$$\rho^2\frac{d^2}{d\rho^2}J_m + \rho\frac{d}{d\rho}J_m + \left(\rho^2 - m^2\right)J_m = 0,$$

and a_m are some coefficients to be determined below.

(c) Given the behaviour

$$J_m(\rho) \to \frac{\rho^m}{2^m m!} \text{ as } \rho \to 0,$$

find the coefficients a_m in the previous part.

(d) Find the expression for the action of $U_\eta U_T$ on the Fourier components c_m of the wavefunction

$$\psi(\theta, t) = \sum_{m=-\infty}^{\infty} c_m(t)e^{im\theta}$$

(e) Show that if

$$\frac{\hbar T}{2I} = 2\pi,$$

then U_T has no effect on the time evolution. In this case, find the expectation value of the kinetic energy of the particle after N steps, assuming it starts in the $m = 0$ ground state.

2 A system consists of two spin-1/2 subsystems, labelled A and B, with states denoted by $|s\rangle_{A,B}$, with $s = \uparrow, \downarrow$ corresponding to eigenvalues of the Pauli matrix $\sigma^z_{A,B}$ equal to ± 1.

A general state of the composite system may be written

$$|\chi\rangle = a_1 |\uparrow\rangle_A |\uparrow\rangle_B + a_2 |\uparrow\rangle_A |\downarrow\rangle_B + a_3 |\downarrow\rangle_A |\uparrow\rangle_B + a_4 |\downarrow\rangle_A |\downarrow\rangle_B$$

(a) Show that the reduced density matrix of the A subsystem $\rho_A \equiv \mathrm{tr}_B |\chi\rangle \langle\chi|$ is

$$\rho_A = \begin{pmatrix} |a_1|^2 + |a_2|^2 & a_1 a_3^* + a_2 a_4^* \\ a_1^* a_3 + a_2^* a_4 & |a_3|^2 + |a_4|^2 \end{pmatrix}.$$

(b) Consider the alternative basis of states

$$\begin{aligned}
|1\rangle &= \frac{i}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B \right) \\
|2\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B \right) \\
|3\rangle &= \frac{i}{\sqrt{2}} \left(|\uparrow\rangle_A |\uparrow\rangle_B - |\downarrow\rangle_A |\downarrow\rangle_B \right) \\
|4\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right).
\end{aligned}$$

Writing the general state now as

$$|\chi\rangle = b_1 |1\rangle + b_2 |2\rangle + b_3 |3\rangle + b_4 |4\rangle ,$$

show that the determinant of ρ_A is

$$\det \rho_A = \frac{1}{4} \Big| \sum_{i=1}^{4} b_i^2 \Big|^2 .$$

Find the eigenvalues of the density matrix and the entanglement entropy $S_A = -\,\mathrm{tr}\,[\rho_A \log \rho_A]$.

(c) What are the conditions satisfied by a matrix belonging to the Lie group $SO(4)$? Explain why the matrices

$$\mathsf{L}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \mathsf{L}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \mathsf{L}_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathsf{K}_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathsf{K}_y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathsf{K}_z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

form a basis for the Lie algebra $so(4)$.

(d) The algebra of these generators is

$$[\mathsf{L}_a, \mathsf{L}_b] = \varepsilon_{abc}\mathsf{L}_c$$
$$[\mathsf{L}_a, \mathsf{K}_b] = \varepsilon_{abc}\mathsf{K}_c\,, \qquad \text{(summation implied)}$$
$$[\mathsf{K}_a, \mathsf{K}_b] = \varepsilon_{abc}\mathsf{L}_c.$$

Find the commutation relations satisfied by $L_a^{\pm} \equiv (L_a \pm K_a)/2$ and interpret this result.

(e) By considering the action of $\sigma_A^{x,y,z}$ on the states in part (b), explain why the entanglement entropy is independent of unitary transformations of the A spin only.

3 Bosons with spin that move between two sites are described by the Hamiltonian

$$H = \overbrace{t\sum_{s=\uparrow,\downarrow}\left(a_s^\dagger b_s + b_s^\dagger a_s\right)}^{H_t} + \mathcal{E}(\Delta)$$

where $a_s^\dagger$, a_s, $b_s^\dagger$, b_s are creation and annihilation operators for the A and B sites, $N_a = \sum_{s=\uparrow,\downarrow} a_s^\dagger a_s$, $N_b = \sum_{s=\uparrow,\downarrow} b_s^\dagger b_s$ and $\mathcal{E}(N_a - N_b)$ is some function of the number difference $\Delta \equiv N_a - N_b$.

(a) Show that

$$\left[a_s^\dagger b_s, \mathcal{E}(\Delta)\right] = \left(\mathcal{E}(\Delta - 2) - \mathcal{E}(\Delta)\right) a_s^\dagger b_s$$
$$\left[b_s^\dagger a_s, \mathcal{E}(\Delta)\right] = \left(\mathcal{E}(\Delta + 2) - \mathcal{E}(\Delta)\right) b_s^\dagger a_s.$$

[Hint: It might be easier to explain why $a_s^\dagger b_s \mathcal{E}(\Delta) = \mathcal{E}(\Delta - 2) a_s^\dagger b_s$*]*

(b) A unitary transformation $H \to H' \equiv e^S H e^{-S}$ is performed to remove H_t from the Hamiltonian at lowest order. Show that S must be chosen so that

$$[S, \mathcal{E}(\Delta)] = -H_t$$

(c) By taking S to have the form

$$S = f(\Delta)\sum_{s=\uparrow,\downarrow} a_s^\dagger b_s - \text{h.c.},$$

(h.c. denotes the hermitian conjugate), find the function $f(\Delta)$.

(d) Show that the transformed Hamiltonian H' contains the term

$$H^{(2)} = \frac{1}{2}\left[S, H_t\right]$$

of order t^2. Evaluate the part of $H^{(2)}$ that does not change the occupancy of the two sites, leaving your answer expressed in terms of $f(\Delta)$.

(e) Discuss how the form of $H^{(2)}$ depends on whether $\mathcal{E}(\Delta)$ is linear (no interactions between particles) or not.

4 The Klein–Gordon equation is

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\Psi(\mathbf{r},t) = 0. \tag{1}$$

(a) Explain some of the difficulties in interpreting $\Psi(\mathbf{r},t)$ as a wavefunction, in contrast to solutions of Schrödinger's equation.

(b) A vector potential is introduced by the replacement $\nabla \to \nabla - ie\mathbf{A}/\hbar$. Show that the equation obeyed by the amplitude $\psi_{\mathbf{k}}(t)$ of a plane wave $\Psi(\mathbf{r},t) = \psi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}}$ in a spatially constant vector potential $\mathbf{A}(t)$ is

$$\ddot{\psi}_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}-e\mathbf{A}(t)/\hbar}\psi_{\mathbf{k}}(t) = 0,$$

where you should state the form of $\omega_{\mathbf{k}}$.

(c) Since $\mathbf{E} = -\dot{\mathbf{A}}$, an electric field pulse at time $t = 0$ is described by a vector potential

$$\mathbf{A}(t) = \begin{cases} 0 & t < 0 \\ \mathbf{A}_0 & t \geq 0 \end{cases}$$

Find the form of $\psi_{\mathbf{k}}(t)$ at $t > 0$ if at $t < 0$

$$\psi_{\mathbf{k}}(t) = \sqrt{\frac{1}{2\omega_{\mathbf{k}}}}a_{\mathbf{k}} \exp\left(-i\omega_{\mathbf{k}}t\right),$$

[*Hint: think of this as a 1D scattering problem*]

(d) By interpreting the positive and negative frequency parts of the $t > 0$ solution in terms of $a_{\mathbf{k}}$ and $b^{\dagger}_{-\mathbf{k}}$, show that the vacuum condition

$$a_{\mathbf{k}}\left|\text{VAC}\right\rangle = 0, \qquad t < 0$$

becomes

$$\left(u_{\mathbf{k}}a_{\mathbf{k}} + v_{\mathbf{k}}b^{\dagger}_{-\mathbf{k}}\right)\left|\text{VAC}\right\rangle = 0, \qquad t > 0,$$

where you should find $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$.

(e) Show that $\left|\text{VAC}\right\rangle$ differs from the state with no $t > 0$ particles or antiparticles by a factor

$$\mathcal{N}\prod_{\mathbf{k}} \exp\left(-(v_{\mathbf{k}}/u_{\mathbf{k}})a^{\dagger}_{\mathbf{k}}b^{\dagger}_{-\mathbf{k}}\right)$$

with some normalization $\mathcal{N}$, and find the probability distribution of $(\mathbf{k}, -\mathbf{k})$ particle-antiparticle pairs in terms of $v_{\mathbf{k}}/u_{\mathbf{k}}$.

END OF PAPER