UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

26 April 2017, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***all four*** *questions. Each question consists of 5 parts, worth 5 marks each. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 In second quantization two-body interactions $U(x,x')$ can be described by

$$\hat{H}_{\text{int}} = \frac{1}{2}\int dx\, dx' \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x')U(x,x')\hat{\psi}(x')\hat{\psi}(x).$$

(a) Which algebraic relations do the operators $\hat{\psi}(x)$ satisfy for bosons (fermions)? Which algebraic relations do the creation and annihilation operators, $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$, in some orthonormal single-particle basis $\{\varphi_\alpha(x)\}$ satisfy for bosons (fermions)? How are the number operators defined? What are their possible eigenvalues for bosons (fermions)?

(b) Consider as the single-particle basis, two orthonormal eigenfunctions of a uniform 1D ring of length L, i.e. $\varphi_1(x) \propto e^{ik_1x}$ and $\varphi_2(x) \propto e^{ik_2x}$. Show that the bosonic Hamiltonian for the interaction potential $U(x,x') = g\delta(x-x')$ is

$$\hat{H}_{\text{int}} = \alpha\hat{n}_1(\hat{n}_1-1) + \beta\hat{n}_2(\hat{n}_2-1) + \gamma\hat{n}_1\hat{n}_2 \qquad (\star)$$

where $\hat{n}_1$ and $\hat{n}_2$ are number operators corresponding to single-particle states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ with real-space wavefunctions $\varphi_1(x)$ and $\varphi_2(x)$, respectively.

(c) Find the ground state of $(\star)$ for a fixed total number, $\hat{n}_1 + \hat{n}_2 = N$, and repulsive interactions $g > 0$. If you have not solved (b), then find the ground state of $(\star)$ for $\alpha = \beta < 0$ and $\gamma = 0$, and a fixed total number, $\hat{n}_1 + \hat{n}_2 = N$.

(d) How would you take into account dispersion, i.e. that the energies of the two single-particle states $\varphi_1(x)$ and $\varphi_2(x)$ could be different? Write down the Hamiltonian including dispersion and its explicit matrix for $N = 2$ bosons. Find the ground state of this Hamiltonian for repulsive interactions $g > 0$.

(e) Consider as the single-particle basis, two orthonormal eigenfunctions of 1D box potential of length L, i.e. $\varphi_1(x) \propto \sin k_1x$ and $\varphi_2(x) \propto \sin k_2x$. Find the Hamiltonian for the interaction potential $U(x,x') = g\delta(x-x')$ as in (b). Which feature complicates your finding of the ground state similar to (c)?

Solution 1. (a) Bookwork. (b) Using the expansion $\hat{\psi}(x) = \varphi_1(x)\hat{a}_1 + \varphi_2(x)\hat{a}_2$ we obtain the Hamiltonian $\hat{H}_{\text{int}} = \frac{1}{2}\sum_{klmn} U_{klnm}a_k^\dagger a_l^\dagger a_m a_n$. The non-zero elements are $U_{1111} = U_{2222} = g$, $U_{1212} = U_{1221} = U_{2112} = U_{2121} = g$ for the single-particle states, so $\hat{H}_{\text{int}} = \frac{g}{2L}\left[\hat{n}_1(\hat{n}_1-1) + \hat{n}_2(\hat{n}_2-1) + 4\hat{n}_1\hat{n}_2\right]$. (c) Using $\hat{n}_2 = N - \hat{n}_1$ we get $\hat{H}_{\text{int}} = \frac{g}{2L}\left[N^2 - N + 2\hat{n}_1(N-\hat{n}_1)\right]$ so ground state twofold degenerate $|N,0\rangle$, $|0,N\rangle$. For $\alpha = \beta < 0$ and $\gamma = 0$, $\hat{H}_{\text{int}} = \alpha\left[N^2 - N - 2\hat{n}_1(N-\hat{n}_1)\right]$ with the same ground states. (d) $\hat{H} = E_1\hat{n}_1 + E_2\hat{n}_2 + \hat{H}_{\text{int}}$ which is diagonal. For $N = 2$, this is a 3×3 matrix. For $E_2 - E_1 > 0$ it is $|N,0\rangle$, for $E_2 - E_1 < 0$ it is $|0,N\rangle$. (e) The overlap integrals can be done using $2\sin^2\frac{x}{2} = 1 - \cos x$ and orthogonality of $\cos mx$. The complication is that $\hat{H}_{\text{int}}$ is not diagonal in this basis: it contains terms $\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_2\hat{a}_2$ with a coefficient $\int_0^L dx\, \sin^2 k_nx \sin^2 k_mx = \frac{L}{4}(1 + \frac{1}{2}\delta_{mn})$.

2 Consider the Hamiltonian of a harmonic oscillator with an external force F

$$H = \frac{p^2}{2M} + \frac{M\omega^2 x^2}{2} - Fx. \qquad (\star\star)$$

(a) Give the eigenstates and the eigenenergies of the Hamiltonian $(\star\star)$.

(b) What is the definition of the coherent state $|\alpha\rangle$? What is its position uncertainty? Show that $|\alpha\rangle$ is the displaced vacuum, i.e. $|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle$, where $|0\rangle$ is the ground state of the oscillator.

Hint: For $[A, B] = \alpha$ we have $e^A e^B = e^{A+B}e^{\alpha/2} = e^B e^A e^\alpha$.

(c) Assuming the system is in the ground state for $F = 0$ at time $t = 0$, when the force is suddenly switched on $F = F_0$, what is the state at the time $t > 0$?

A particle with spin $\frac{1}{2}$ is in thermal equilibrium with a thermal bath at absolute temperature T and under the influence of constant magnetic field $\mathbf{B}$.

(d) Write down the density operator ϱ describing the spin $\frac{1}{2}$ in this situation. In which limit is the state of the system pure? When is it maximally mixed? In which situation would the notion of negative temperatures make sense?

(e) Calculate the expectation value of the spin component $\hat{S}_z$ for $\mathbf{B} = B\mathbf{e}_z$. Sketch the dependence of your result as a function of (inverse) temperature.

Solution 2. (a) Completing the square $H = \frac{p^2}{2M} + \frac{M\omega^2}{2}(x - x_0)^2 - \frac{F_0^2}{2M\omega^2}$, we find the eigenstates are displaced oscillator states with $E_n = \hbar\omega n - \frac{F_0^2}{2M\omega^2}$ and $x_0 = \frac{F_0}{m\omega^2}$. (b) Bookwork. $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. $\langle\alpha|\hat{x}^2|\alpha\rangle - \langle\alpha|\hat{x}|\alpha\rangle^2 = x_{\mathrm{ZPF}}^2$. (c) From part (a), we know that the initial state is a coherent state $|\psi(t = 0)\rangle = |\alpha_0\rangle$ with $\alpha_0 = \frac{F_0 x_{\mathrm{ZPF}}}{\hbar\omega}$ so the dynamics is $|\psi(t)\rangle = |e^{-i\omega t}\alpha_0\rangle$. (d) $\varrho = \frac{1}{Z}e^{-\beta H}$ and $Z = \mathrm{tr}\,\varrho$ and $H = -\gamma\mathbf{B}\cdot\mathbf{S}$. Negative temperatures correspond the spin pointing in the opposite direction. (e) $\langle\hat{S}_z\rangle = \mathrm{tr}\,\hat{S}_z\varrho = \frac{\hbar}{2}\frac{e^{-\gamma\beta B} - e^{+\gamma\beta B}}{e^{-\gamma\beta B} + e^{+\gamma\beta B}}$, i.e. pure state $T \to 0$ and totally mixed $T \to \infty$.

3

(a) Show that the propagator of a one-dimensional time-independent Hamiltonian with a complete set of energy eigenfunctions $\{\varphi_\alpha(x)\}$ and eigenvalues $\{E_\alpha\}$ is

$$K(x,t|x',t') = \theta(t-t')\sum_\alpha \varphi_\alpha(x)\varphi_\alpha^*(x')e^{-iE_\alpha(t-t')/\hbar}.$$

(b) The Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - g\delta(x)$$

describes an attractive δ-function potential ($g>0$). There is a single bound state of the form

$$\varphi_0(x) = \sqrt{\kappa}\, e^{-\kappa|x|}.$$

Find κ and the energy E_0 of the bound state.

(c) Find the phase shifts $\delta_{\text{even}}(k)$ and $\delta_{\text{odd}}(k)$ for scattering in the even and odd channels at wavevector k.

(d) Show that the propagator $K(0,T|0,0)$ can be written

$$K(0,T|0,0) = \kappa e^{-iE_0T/\hbar} + \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{k^2}{\kappa^2+k^2}\exp\left(-\frac{i\hbar k^2 T}{2m}\right). \qquad (\star\star\star)$$

(e) How does the expression $(\star\star\star)$ behave as $T\to\infty$?

Solution 3.

(a) The propagator is

$$K(x,t|x',t') = \theta(t-t')\,\langle x|U(t,t')|x'\rangle\,.$$

Writing

$$U(t,t') = e^{-iH(t-t')/\hbar} = \sum_\alpha |\varphi_\alpha\rangle\,\langle\varphi_\alpha|\,e^{-iE_\alpha(t-t')/\hbar}$$

and taking matrix elements gives the answer.

(b) This is a question of subbing in

$$\partial_x^2\varphi_0(x) = \kappa^2\varphi_0(x) + 2\kappa\delta(x),$$

from which we get $\kappa = gm/\hbar^2$ and $E_0 = \hbar^2\kappa^2/2m$.

(c) $\delta_{\mathrm{odd}}(k) = 0$ as the odd wavefunctions vanish at the origin.

Write the even wavefunctions as

$$\psi_{k,\mathrm{even}}(x) = \cos(k|x| - \delta_{\mathrm{even}}(k)).$$

Apply the boundary condition

$$\frac{\hbar^2}{2m}\psi'(x)|_-^+ = -g\psi(0)$$

gives

$$\frac{\hbar^2 k}{m}\sin\delta_{\mathrm{even}} = g\cos\delta_{\mathrm{even}}$$

or

$$\tan\delta_{\mathrm{even}} = \kappa/k.$$

(d) This is a matter of substituting the results of (b) and (c) into the formula from part (a). The odd channel doesn't contribute. Each state in the even channel contributes

$$\cos^2\delta_{\mathrm{even}} = \frac{k^2}{k^2+\kappa^2}.$$

The correct numerical factor requires a density of states calculation. Alternatively, the shortcut is to note that $\kappa \to 0$ should recover the free particle propagator, which fixes the factor.

(e) The best way to get at this is write the second term as

$$\left(\frac{m}{\hbar^2 T}\right)^{3/2}\int_{-\infty}^{\infty}\frac{du}{2\pi}\frac{u^2}{\kappa^2 + mu^2/(\hbar^2 T)}\exp\left(-\frac{iu^2}{2}\right).$$

Expanding the denominator and doing the integrals yields a series (actually an asymptotic series) in $T^{-m-1/2}$ for $m = 1, 2, \ldots$. The behaviour at long times is thus dominated by the term $\kappa e^{-iE_0T/\hbar}$ arising from the bound state.

4 Consider the scattering problem in two dimensions.

(a) Show that the asymptotic form of the scattered wave in this case is

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} e^{ikr\cos\theta} + \sqrt{\frac{i}{kr}} f(\theta) e^{ikr},$$

which defines the (dimensionless) scattering amplitude $f(\theta)$ in two dimensions.

[*Hint: You may find it helpful to know that the form of the Laplacian in polar coordinates* (r, θ) *is*

$$\nabla^2 f = \frac{1}{r}\partial_r \left(r\partial_r f\right) + \frac{1}{r^2}\partial_\theta^2 f.$$
]

(b) Show that the total cross-section, which has the units of length, is

$$\lambda = \frac{1}{k}\int d\theta |f(\theta)|^2.$$

(c) Show that the 2D plane wave has the expansion

$$\exp\left(ikr\cos\theta\right) = \sum_{m=-\infty}^{\infty} a_m J_m(kr) e^{im\theta},$$

where $J_m(\rho)$ is the Bessel function satisfying

$$\rho^2 \frac{d^2}{d\rho^2} J_m + \rho \frac{d}{d\rho} J_m + \left(\rho^2 - m^2\right) J_m = 0,$$

and a_m are some coefficients to be determined below.

(d) Given the behaviour

$$J_m(\rho) \to \frac{\rho^m}{2^m m!} \text{ as } \rho \to 0,$$

find the coefficients a_m in the previous part.

(e) The asymptotic form of the scattered wave can be written

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \sum_{m=0}^{\infty} \sqrt{\frac{2}{\pi kr}} \epsilon_m i^m e^{i\delta_m} \cos(m\theta) \cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4} + \delta_m\right)$$

where $\epsilon_0 = 2$ and $\epsilon_{m\neq 0} = 1$. Find expressions for $f(\theta)$ and λ in terms of δ_m.

Solution 4.

(a) The given asymptotic form satisfies

$$\left(\nabla^2 + k^2\right) \psi_{\mathbf{k}} = \sqrt{\frac{i}{k}} \left(\frac{f(\theta)}{4} + f''(\theta)\right) \frac{e^{ikr}}{r^{5/2}}.$$

The remainder on the RHS will be cancelled by subleading terms in the asymptotic expansion.

(b) The non-oscillating terms in the flux

$$j = -\frac{i\hbar}{2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

are

$$\frac{\hbar k}{m} \hat{\mathbf{x}} + \frac{\hbar |f(\theta)|^2}{mr} \hat{\mathbf{r}}.$$

The probability / time into an angle $d\theta$ is $\propto (\hbar/m)|f(\theta)|^2 d\theta$ while the incident flux is $\propto \hbar k/m$. The cross section is the ratio

$$d\lambda = \frac{|f(\theta)|^2 d\theta}{k}.$$

Integrating over θ gives the total cross section.

(c) We look for separable solutions of the Helmholtz equation

$$\left[\nabla^2 + k^2\right] \Phi(r, \theta) = 0,$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

Separable solutions are then shown to have the form $\Phi(r, \theta) = J_m(kr) e^{im\theta}$. The expansion of the plane wave then follows.

(d) This follows closely the 3D case. The key point is that a term $(kr)^m e^{im\theta}$ arises only from the m^{th} order. Using the given asymptote and comparing coefficients then gives $a_m = i^m$.

(e) We have

$$i^m e^{i\delta_m} \cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4} + \delta_m\right) = \frac{1}{2}(\overbrace{\sqrt{-i} e^{ikr + 2\delta_m}}^{\text{out}} + \overbrace{\sqrt{i}(-1)^m e^{-ikr}}^{\text{in}}). \quad (1)$$

The crucial part is to subtract the asymptotic form of the plane wave expansion, which corresponds to setting $\delta_m = 0$, to leave only the outgoing wave. The scattering amplitude $f(\theta)$ can then be read off as

$$f(\theta) = \sum_{m=0}^{\infty} \sqrt{\frac{2}{\pi}} \epsilon_m e^{i\delta_m} \sin \delta_m \cos(m\theta).$$

Evaluating λ gives

$$\lambda = \frac{4}{k} \sum_{m=0}^{\infty} \epsilon_m \sin^2 \delta_m.$$

END OF PAPER