NATURAL SCIENCES TRIPOS Part II

26 April 2017, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***all four*** *questions. Each question consists of 5 parts, worth 5 marks each. The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

1 In second quantization two-body interactions $U(x, x')$ can be described by

$$\hat{H}_{\text{int}} = \frac{1}{2} \int dx\, dx' \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') U(x, x') \hat{\psi}(x') \hat{\psi}(x).$$

(a) Which algebraic relations do the operators $\hat{\psi}(x)$ satisfy for bosons (fermions)? Which algebraic relations do the creation and annihilation operators, $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$, in some orthonormal single-particle basis $\{\varphi_\alpha(x)\}$ satisfy for bosons (fermions)? How are the number operators defined? What are their possible eigenvalues for bosons (fermions)?

(b) Consider as the single-particle basis, two orthonormal eigenfunctions of a uniform 1D ring of length L, i.e. $\varphi_1(x) \propto e^{ik_1 x}$ and $\varphi_2(x) \propto e^{ik_2 x}$. Show that the bosonic Hamiltonian for the interaction potential $U(x, x') = g\delta(x - x')$ is

$$\hat{H}_{\text{int}} = \alpha \hat{n}_1(\hat{n}_1 - 1) + \beta \hat{n}_2(\hat{n}_2 - 1) + \gamma \hat{n}_1 \hat{n}_2 \qquad (\star)$$

where $\hat{n}_1$ and $\hat{n}_2$ are number operators corresponding to single-particle states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ with real-space wavefunctions $\varphi_1(x)$ and $\varphi_2(x)$, respectively.

(c) Find the ground state of $(\star)$ for a fixed total number, $\hat{n}_1 + \hat{n}_2 = N$, and repulsive interactions $g > 0$. If you have not solved (b), then find the ground state of $(\star)$ for $\alpha = \beta < 0$ and $\gamma = 0$, and an even total number, $\hat{n}_1 + \hat{n}_2 = N$.

(d) How would you take into account dispersion, i.e. that the energies of the two single-particle states $\varphi_1(x)$ and $\varphi_2(x)$ could be different? Write down the Hamiltonian including dispersion and its explicit matrix for $N = 2$ bosons. Find the ground state of this Hamiltonian for repulsive interactions $g > 0$.

(e) Consider as the single-particle basis, two orthonormal eigenfunctions of 1D box potential of length L, i.e. $\varphi_1(x) \propto \sin k_1 x$ and $\varphi_2(x) \propto \sin k_2 x$. Find the Hamiltonian for the interaction potential $U(x, x') = g\delta(x - x')$ as in (b). Which feature complicates your finding of the ground state similar to (c)?

2 Consider the Hamiltonian of a harmonic oscillator with an external force F

$$H = \frac{p^2}{2M} + \frac{M\omega^2 x^2}{2} - Fx. \qquad (\star\star)$$

(a) Give the eigenstates and the eigenenergies of the Hamiltonian $(\star\star)$.

(b) What is the definition of the coherent state $|\alpha\rangle$? What is its position uncertainty? Show that $|\alpha\rangle$ is the displaced vacuum, i.e. $|\alpha\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\,|0\rangle$, where $|0\rangle$ is the ground state of the oscillator.

Hint: For $[A, B] = \alpha$ we have $e^A e^B = e^{A+B} e^{\alpha/2} = e^B e^A e^\alpha$.

(c) Assuming the system is in the ground state for $F = 0$ at time $t = 0$, when the force is suddenly switched on $F = F_0$, what is the state at the time $t > 0$?

A particle with spin $\frac{1}{2}$ is in thermal equilibrium with a thermal bath at absolute temperature T and under the influence of constant magnetic field $\mathbf{B}$.

(d) Write down the density operator ϱ describing the spin $\frac{1}{2}$ in this situation. In which limit is the state of the system pure? When is it maximally mixed? In which situation would the notion of negative temperatures make sense?

(e) Calculate the expectation value of the spin component $\hat{S}_z$ for $\mathbf{B} = B\mathbf{e}_z$. Sketch the dependence of your result as a function of (inverse) temperature.

3

(a) Show that the propagator of a one-dimensional time-independent Hamiltonian with a complete set of energy eigenfunctions $\{\varphi_\alpha(x)\}$ and eigenvalues $\{E_\alpha\}$ is

$$K(x,t|x',t') = \theta(t-t')\sum_\alpha \varphi_\alpha(x)\varphi^*_\alpha(x')e^{-iE_\alpha(t-t')/\hbar}.$$

(b) The Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - g\delta(x)$$

describes an attractive δ-function potential ($g > 0$). There is a single bound state of the form

$$\varphi_0(x) = \sqrt{\kappa}\, e^{-\kappa|x|}.$$

Find κ and the energy E_0 of the bound state.

(c) Find the phase shifts $\delta_{\text{even}}(k)$ and $\delta_{\text{odd}}(k)$ for scattering in the even and odd channels at wavevector k.

(d) Show that the propagator $K(0,T|0,0)$ can be written

$$K(0,T|0,0) = \kappa e^{-iE_0T/\hbar} + \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{k^2}{\kappa^2+k^2}\exp\left(-\frac{i\hbar k^2 T}{2m}\right). \qquad (\star\star\star)$$

(e) How does the expression $(\star\star\star)$ behave as $T\to\infty$?

4 Consider the scattering problem in two dimensions.

(a) Show that the asymptotic form of the scattered wave in this case is

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} e^{ikr\cos\theta} + \sqrt{\frac{i}{kr}} f(\theta) e^{ikr},$$

which defines the (dimensionless) scattering amplitude $f(\theta)$ in two dimensions.

[*Hint: You may find it helpful to know that the form of the Laplacian in polar coordinates* (r,θ) *is*

$$\nabla^2 f = \frac{1}{r}\partial_r\left(r\partial_r f\right) + \frac{1}{r^2}\partial_\theta^2 f.$$
]

(b) Show that the total cross-section, which has the units of length, is

$$\lambda = \frac{1}{k}\int d\theta |f(\theta)|^2.$$

(c) Show that the 2D plane wave has the expansion

$$\exp\left(ikr\cos\theta\right) = \sum_{m=-\infty}^{\infty} a_m J_m(kr) e^{im\theta},$$

where $J_m(\rho)$ is the Bessel function satisfying

$$\rho^2 \frac{d^2}{d\rho^2} J_m + \rho \frac{d}{d\rho} J_m + \left(\rho^2 - m^2\right) J_m = 0,$$

and a_m are some coefficients to be determined below.

(d) Given the behaviour

$$J_m(\rho) \to \frac{\rho^m}{2^m m!} \text{ as } \rho \to 0,$$

find the coefficients a_m in the previous part.

(e) The asymptotic form of the scattered wave can be written

$$\psi_{\mathbf{k}}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \sum_{m=0}^{\infty} \sqrt{\frac{2}{\pi kr}} \epsilon_m i^m e^{i\delta_m} \cos(m\theta) \cos\left(kr - \frac{m\pi}{2} - \frac{\pi}{4} + \delta_m\right)$$

where $\epsilon_0 = 2$ and $\epsilon_{m\neq 0} = 1$. Find expressions for $f(\theta)$ and λ in terms of δ_m.

END OF PAPER