UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

25 April 2018, 9.00 to 11.00

THEORETICAL PHYSICS 2

*Answer **all four** questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 Consider the following Hamiltonian where $\hat{c}_1$ and $\hat{c}_2$ are (spinless) fermions

$$H = -\left[t\left(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1\right) + \Delta\left(\hat{c}_1^\dagger \hat{c}_2^\dagger + \hat{c}_2 \hat{c}_1\right)\right].$$

(a) Which algebraic relations do creation and annihilation operators satisfy for bosons (fermions)? How are number operators defined? What are their possible eigenvalues for bosons (fermions)? [5]

(b) Find the anti-commutation relations for the *Majorana* operators $j = 1, 2$

$$\gamma_j^A = \hat{c}_j + \hat{c}_j^\dagger \qquad \gamma_j^B = -i(\hat{c}_j - \hat{c}_j^\dagger).$$

Discuss if it is possible to define occupation numbers for *Majorana* fermions. [5]

(c) For $t = \Delta \neq 0$ show that the Hamiltonian given above can be written as

$$H = it\gamma_1^B \gamma_2^A.$$

Find the time dependence (in the Heisenberg picture) for the two *Majorana* operators that are not featured in this Hamiltonian, i.e. $\gamma_1^A(t)$ and $\gamma_2^B(t)$. [5]

(d) Verify the following operators have fermionic anti-commutation relations [4]

$$\hat{f}_1 = \tfrac{1}{2}\left(\gamma_1^A + i\gamma_2^B\right) \qquad \hat{f}_2 = \tfrac{1}{2}\left(\gamma_1^B + i\gamma_2^A\right).$$

(e) For $t = \Delta \neq 0$ rewrite the Hamiltonian in (c) in terms of the operators $\hat{f}_1$ and $\hat{f}_2$. Determine the eigenvalues of the Hamiltonian and their degeneracy. [6]

Solution:

(a) Bookwork:

Algebra: (anti)commutation relations for creation/annihilation operators;

Number operators: $n_\alpha = a_\alpha^\dagger a_\alpha$

Eigenvalues: $0, 1 \ldots \infty$ for bosons, $0, 1$ for fermions.

(b) By expanding in terms of the original fermionic operators, one gets $\{\gamma_j^a, \gamma_k^b\} = 2\delta_{ij}\delta_{ab}$.

(c) is straightforward, since both γ_1^A and γ_2^B commute with the Hamiltonian (and therefore $\partial_t \gamma_1^A = \partial_t \gamma_2^B = 0$. They also commute with $e^{-iHt/\hbar}$, if preferring the direct evolution version (this can be seen expanding the exponential, and all terms are pairs of Majoranas that are different from the one being considered). Hence,
$\gamma_1^A(t) = \gamma_1^A(0)$ and $\gamma_2^B(t) = \gamma_2^B(0)$.

(d) Anticommutation: again, expand and get it.

(e) The Hamiltonian becomes $H = t(2\hat{f}_2^\dagger \hat{f}_2 - 1)$. All the eigenenergies $-t$ and t are doubly degenerate ($\hat{f}_1$ is a zero mode, the energy does not change if occupied or not).

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2 Consider scattering from a so-called *delta-shell potential* $V(\mathbf{r}) = V_0\delta(r-R)$.

(a) Write down the asymptotic solution to the 3D scattering problem. What is the scattering amplitude, the differential and total scattering cross section? In what sense is this an asymptotic solution rather than an exact solution? [5]

(b) Calculate the scattering amplitude for the delta-shell potential as given above in the first Born approximation. [5]

(c) Starting from the differential equation with boundary condition $u_0(0) = 0$

$$u_0''(r) + k^2 u_0(r) = \frac{2m}{\hbar^2} V(r) u_0(r),$$

find the phase shift $\delta_0(k)$ for s-wave scattering as a function of wave vector k

$$k\cot(kR+\delta_0) - k\cot kR = \frac{2mV_0}{\hbar^2}$$

for a particle with mass m. [5]

(d) Determine from the result in (c) the scattering length a_0 as a function of V_0. Sketch the function $a_0(V_0)$, choosing the appropriate dimensionless units, both in the repulsive and attractive case, and discuss its prominent features. [5]

(e) Compare the scattering amplitude obtained in the Born approximation in (b) and in partial wave analysis in (c) and (d), which are given also here,

$$f_{\text{Born}} = -\frac{2mV_0R\sin qR}{\hbar^2 q} \qquad f_{s\text{-wave}} = -\frac{\frac{2mV_0R}{\hbar^2}R}{1+\frac{2mV_0R}{\hbar^2}}.$$

In which aspects do they differ? In which limit do you expect them to agree? Show that they indeed lead to the same expression in that limit. [5]

Solution 2. (a) Bookwork. (b) Starting $f(\theta) = -\frac{m}{2\pi\hbar^2}\int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r})$, performing the integration over angles $f(\theta) = -\frac{2m}{\hbar^2 q}\int dr rV(r)\sin qr = -\frac{2mV_0R}{\hbar^2 q}\sin qR$. (c) Use ansatz $u_0(r) = rR_0(r) = A\sin kr$ for $r<R$ and $u_0(r) = B\sin(kr+\delta_0)$ for $r>R$, matching $u_0(r)$ at $r=R$ and with $\epsilon>0$ have $u_0'(R+\epsilon) - u_0'(R-\epsilon) = \frac{2mV_0}{\hbar^2}u(R)$. (d) Use $\tan kR \approx kR$ and $a_0 = -\lim_{k\to 0}\frac{\delta_0}{k}$ to get $\frac{a_0}{R} = \frac{\beta}{1+\beta}$ with $\beta = \frac{2mV_0R}{\hbar^2}$, discuss hard sphere, zero scattering, resonance. (e) f_{Born} is anisotropic, perturbative in V_0. We need to take the limit $kR \ll 1$. $f_{s\text{-wave}}$ is isotropic and non-perturbative in V_0, so we need to expand in powers of V_0. The two expressions give $f(\theta) = -\frac{2mV_0R^2}{\hbar^2}$.

3

(a) Define the density matrix ρ. Show that ρ is hermitian and that $\mathrm{tr}[\rho] = 1$. State the condition on $\mathrm{tr}[\rho^2]$ for pure and mixed states. [5]

(b) Consider the following Hamiltonian

$$H = -t\left(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1\right)$$

describing spinless fermions that can move between two sites, where $\hat{c}_1^\dagger$, $\hat{c}_1$, $\hat{c}_2^\dagger$, and $\hat{c}_2$ are the creation and annihilation operators for the sites 1 and 2.

For $N = \hat{c}_1^\dagger \hat{c}_1 + \hat{c}_2^\dagger \hat{c}_2 = 1$ determine the density matrix describing this system in equilibrium with a thermal reservoir at temperature T. [5]

Obtain the $T = 0$ and $T \to \infty$ limits and check the properties stated in part (a). [3]

Obtain the von Neumann entropy of the system at any temperature and the low and high temperature limits. [2]

(c) Consider now that the one spinless fermion in the system above (C) is coupled to another similar fermion in another two-state system (called B) described by operators $b_1^\dagger$, b_1, $b_2^\dagger$, and b_2. If the state of the global system is

$$|\Psi\rangle = \left(\frac{1}{2} b_1^\dagger c_1^\dagger + \frac{1}{\sqrt{2}} b_1^\dagger c_2^\dagger + \frac{1}{2} b_2^\dagger c_2^\dagger\right)|0\rangle,$$

where $|0\rangle$ is the vacuum state, obtain the global density matrix. [5]

Compute the reduced density matrix for system C, and indicate how you would compute its entanglement entropy. (This calculation is *not* required.) [5]

Solution:

(a) Bookwork. Definition as in eqs. 4.6 of the lecture notes (page 58): operator that is hermitian, of unit trace and positive semi-definite ($\langle\Psi|\rho|\Psi\rangle \geq 0$), such that $\overline{\langle\mathcal{O}\rangle} = \mathrm{tr}[\rho\mathcal{O}]$. Equivalently, through equation 4.8

$$\rho = \sum_\alpha P_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$$

being $\{|\phi_\alpha\rangle\}$ an orthonormal Hilbert-space basis, and $P_\alpha \geq 0$, and derive the conditions.

$\mathrm{tr}[\rho^2] = 1$ for a pure state, $\mathrm{tr}[\rho^2] < 1$ for mixed.

(b) The Hamiltonian eigenstates are

$$|g\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$
$$|u\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

with eigenvalues $\epsilon_g = -t$ and $\epsilon_u = +t$. The density matrix in the eigenstate basis is

$$\rho = \frac{1}{\mathcal{Z}}\begin{pmatrix} e^{\beta t} & 0 \\ 0 & e^{-\beta t}\end{pmatrix}$$

or $\rho = \left(|g\rangle e^{\beta t}\langle g| + |u\rangle e^{-\beta t}\langle u|\right)/\mathcal{Z}$, where $\mathcal{Z} = e^{\beta t} + e^{-\beta t} = 2\cosh(\beta t)$. Introducing the above expressions for $|g\rangle$ and $|u\rangle$,

$$\begin{aligned}\rho &= \frac{e^{\beta t}}{2\mathcal{Z}}(|1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 2|) + \frac{e^{-\beta t}}{2\mathcal{Z}}(|1\rangle\langle 1| - |1\rangle\langle 2| - |2\rangle\langle 1| + |2\rangle\langle 2|) \\ &= \frac{1}{\mathcal{Z}}\{\cosh(\beta t)(|1\rangle\langle 1| + |2\rangle\langle 2|) + \sinh(\beta t)(|1\rangle\langle 2| + |2\rangle\langle 1|)\}\end{aligned}$$

A

$$\rho = \frac{1}{2\cosh(\beta t)}\begin{pmatrix}\cosh(\beta t) & \sinh(\beta t) \\ \sinh(\beta t) & \cosh(\beta t)\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \tanh(\beta t) \\ \tanh(\beta t) & 1\end{pmatrix}$$

$$T \to 0 \quad \rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} \quad ; \quad T \to \infty \quad \rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

both being hermitian and of trace $= 1$. Since

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

the trace of ρ^2 is 1 for $T = 0$ (a pure state). For $T \to \infty$, $\rho^2 = (1/4)\mathbb{1}$ and therefore $\mathrm{tr}\,\rho = 1/2$, a mixed state.

The von Neumann entropy

$$\begin{aligned} S/k &= -\,\mathrm{tr}(\rho \ln \rho) = -\sum_n P_n \ln P_n = -\left\{ \frac{e^{\beta t}}{\mathcal{Z}}(\beta t - \log \mathcal{Z}) + \frac{e^{-\beta t}}{\mathcal{Z}}(-\beta t - \log \mathcal{Z}) \right\} \\ &= -\frac{1}{2\cosh \beta t}\left\{ \beta t (e^{\beta t} - e^{-\beta t}) - (e^{\beta t} + e^{-\beta t}) \log \mathcal{Z} \right\} = \log \mathcal{Z} - \beta t \tanh \beta t \ , \end{aligned}$$

$S/k \to 0$ for $T \to 0$, since $\mathcal{Z} = 2\cosh(\beta t) \to e^{\beta t}$, and $\tanh \beta t \to 1$.

$S/k \to \log 2$ for $T \to \infty$, since $\mathcal{Z} \to 2$.

(c) In general, for a given (pure) state $|\Psi$ of the global BC system,

$$|\Psi\rangle = \left(\alpha_{11} b_1^\dagger c_1^\dagger + \alpha_{12} b_1^\dagger c_2^\dagger + \alpha_{21} b_2^\dagger c_1^\dagger + \alpha_{22} b_2^\dagger c_2^\dagger \right) |0\rangle,$$

the corresponding density matrix $\rho = |\Psi\rangle\langle\Psi|$ can be written in matrix form, in the basis given by $|1,1\rangle (= b_1^\dagger c_1^\dagger |0\rangle)$, $|1,2\rangle$, $|2,1\rangle$, and $|2,2\rangle$, as

$$\rho_t = \begin{pmatrix} |\alpha_{11}|^2 & \alpha_{11}\alpha_{12}^* & \alpha_{11}\alpha_{21}^* & \alpha_{11}\alpha_{22}^* \\ \alpha_{12}\alpha_{11}^* & |\alpha_{12}|^2 & \alpha_{12}\alpha_{21}^* & \alpha_{12}\alpha_{22}^* \\ \alpha_{21}\alpha_{11}^* & \alpha_{21}\alpha_{12}^* & |\alpha_{21}|^2 & \alpha_{21}\alpha_{22}^* \\ \alpha_{22}\alpha_{11}^* & \alpha_{22}\alpha_{12}^* & \alpha_{22}\alpha_{21}^* & |\alpha_{22}|^2 \end{pmatrix} .$$

The reduced density matrix of the c system

$$(\rho_c^{red})_{\mu\nu} = \sum_i \langle i\mu|\hat{\rho}|i\nu\rangle$$

where i runs over states in system B, and μ, ν over the ones in system C, or

$$\rho_c^{red} = \begin{pmatrix} |\alpha_{11}|^2 + |\alpha_{21}|^2 & \alpha_{11}\alpha_{12}^* + \alpha_{21}\alpha_{22}^* \\ \alpha_{12}\alpha_{11}^* + \alpha_{22}\alpha_{21}^* & |\alpha_{12}|^2 + |\alpha_{22}|^2 \end{pmatrix} .$$

In the case above

$$\rho_c^{red} = \begin{pmatrix} 1/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 3/4 \end{pmatrix} ,$$

which is an impure state ($\mathrm{tr}(\rho^2) < 1$).

The entanglement entropy for system C would be

$$S_c^{ent} = -\,\mathrm{tr}\{\rho_c^{red} \log \rho_c^{red}\} = -\sum_n P_n^c \log P_n^c,$$

where P_n^c are the eigenvalues of the reduced density matrix for the system C, ρ_c^{red}. The procedure would then involve diagonalising ρ_c^{red} and computing the entropy from the eigenvalues.

A

4 Given a harmonic oscillator with associated Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

where m is the mass of the oscillator and ω its angular frequency, and where x and p are the position and linear-momentum operators.

(a) State the equations of motion for the operators $x(t)$ and $p(t)$ in the Heisenberg picture and write down their general solution. Define the ladder operators, a and $a^\dagger$ (annihilation and creation operators), and obtain their time dependence in the Heisenberg picture. [5]

(b) Consider that the Hamiltonian is suddenly modified at $t = 0$ to

$$H = H_0 + \gamma x$$

i.e. displacing both the rest position of the oscillator and the minimum value of the harmonic potential. If the oscillator was in its ground state $|0\rangle$ prior to the change, compute the expectation value of the energy after the change with respect to the new potential energy minimum. Stating the expectation values of the position (with respect to the new minimum) and momentum operators just after the change. Using the solutions for $x(t)$ and $p(t)$ of part (a), write down $\langle x(t)\rangle$ and $\langle p(t)\rangle$. [5]

(c) Using the fact that the Hamiltonian eigenstates after the displacement relate to the old ones through a translation operator, $|n'\rangle = e^{-ipx_0/\hbar}|n\rangle$, and using the following relation,

$$e^A e^B = e^{A+B} e^{[A,B]/2} \ ,$$

derive the following expansion of the initial state $|\Psi(0)\rangle = |0\rangle$ in terms of new $|n'\rangle$ states just after the displacement [10]

$$|\Psi\rangle = \sum_{n'} \frac{1}{\sqrt{n!}} \left(\frac{m\omega x_0^2}{2\hbar}\right)^{n/2} e^{-\frac{m\omega x_0^2}{4\hbar}} |n'\rangle.$$

(d) Show that $|\Psi\rangle$ is an eigenstate of the annihilation operator for the displaced oscillator,

$$a'|\Psi\rangle = \alpha|\Psi\rangle$$

and that it is so for any time $t > 0$. State the eigenvalue α. [5]

Solution 4. Displaced oscillator.

(a) Bookwork, Eq. 1.15 and 1.18 of the lecture notes.

Equations of motion and solutions for $x(t)$ and $p(t)$:

$\partial_t x(t) = p/m$ and $\partial_t p(t) = -m\omega^2 x$; then, $x(t) = \cos(\omega t)x(0) + \sin(\omega t)\frac{p(0)}{m\omega}$, $p(t) = \cos(\omega t)p(0) - m\omega\sin(\omega t)x(0)$.

For the ladder operators: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right)$, the Heisenberg time evolution is: $a(t) = e^{-i\omega t}a(0)$ and $a^\dagger(t) = e^{i\omega t}a^\dagger(0)$.

(b) $\langle H\rangle = \langle 0|H_0|0\rangle + \gamma\langle 0|x|0\rangle = \hbar\omega/2 + \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a + a^\dagger)|0\rangle$. The last term is zero, and therefore the sudden change in potential has not changed the total energy on average.

Displaced oscillator: new $V(x) = \frac{1}{2}m\omega^2(x - x_0)^2 - \frac{1}{2}m\omega^2 x_0^2$, with $x_0 = -\frac{\gamma}{m\omega^2}$. Just after the quench ($t = 0$) and with respect to the new potential energy minimum, $\langle x(0)\rangle = -x_0$ and $\langle p(0)\rangle = 0$, and therefore, $\langle x(t)\rangle = -x_0\cos(\omega t)$ and $\langle p(t)\rangle = x_0 m\omega\sin(\omega t)$.

(c)We need to compute the coefficients of the expansion of the ground state of the old oscillator in terms of the states of the new one, $\langle n'|0\rangle = \langle n|e^{-ip|x_0|/\hbar}|0\rangle$ (since the hermitian conjugate of a translation is a translation with opposite sign, and x_0 above was negative). $p = i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger)$, and then $e^{-ip|x_0|/\hbar} = e^{-\theta(a-a^\dagger)}$, with $\theta = \sqrt{\frac{m\omega}{2\hbar}x_0^2}$. Using the relation offered in the question, $e^{-\theta(a-a^\dagger)} = e^{\theta a^\dagger}e^{-\theta a}e^{-\theta^2/2}$.

The coefficients of the expansion $\langle n'|0\rangle = \langle n|e^{\theta a^\dagger}e^{-\theta a}e^{-\theta^2/2}|0\rangle$.

But $e^{-\theta a}|0\rangle = |0\rangle$, and

$$e^{\theta a^\dagger}|0\rangle = \sum_m \frac{\theta^m}{m!}(a^\dagger)^m|0\rangle = \sum_m \frac{\theta^m}{m!}\sqrt{m!}\,|m\rangle$$

where we have used $a^\dagger|m\rangle = \sqrt{m+1}\,|m+1\rangle$. Therefore,

$$\langle n'|0\rangle = e^{-\theta^2/2}\langle n|\left(\sum_m \frac{\theta^m}{\sqrt{m!}}|m\rangle\right) = \frac{\theta^n}{\sqrt{n!}}e^{-\theta^2/2} .$$

Substituting for θ and getting it into $|\Psi(0)\rangle = |0\rangle = \sum_n |n'\rangle\langle n'|0\rangle$ gives the sought result.

(d)

$$a'|\Psi\rangle = a'\left(\sum_{n=0}\frac{e^{-\theta^2/2}}{\sqrt{n!}}\theta^n|n'\rangle\right) = \sum_{n=0}\frac{e^{-\theta^2/2}}{\sqrt{n!}}\theta^n a'|n'\rangle = \sum_{n=1}\frac{e^{-\theta^2/2}}{\sqrt{n!}}\theta^n\sqrt{n}|(n-1)'\rangle$$
$$= \theta\sum_{n=1}\frac{e^{-\theta^2/2}}{\sqrt{(n-1)!}}\theta^{n-1}|(n-1)'\rangle = \theta|\Psi\rangle.$$

Therefore, $\alpha = \theta = \sqrt{\frac{m\omega}{2\hbar}}|x_0|$. It can be seen that $|\Psi(t)\rangle$ remains an eigenstate of a' at any future time, $a'|\Psi(t)\rangle = \alpha(t)|\Psi(t)\rangle$: turning to the Heisenberg picture, $a'(t)|\Psi(0)\rangle = \alpha(t)|\Psi(0)\rangle$, remembering that $a'(t) = a'e^{-i\omega t}$ (and therefore $\alpha(t) = \theta e^{-i\omega t}$).
