UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

25 April 2018, 9.00 to 11.00

THEORETICAL PHYSICS 2

Answer ***all four*** *questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V7.1

A

1 Consider the following Hamiltonian where $\hat{c}_1$ and $\hat{c}_2$ are (spinless) fermions

$$H = -\left[t\left(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1\right) + \mathit{\Delta}\left(\hat{c}_1^\dagger \hat{c}_2^\dagger + \hat{c}_2 \hat{c}_1\right)\right].$$

(a) Which algebraic relations do creation and annihilation operators satisfy for bosons (fermions)? How are number operators defined? What are their possible eigenvalues for bosons (fermions)? [5]

(b) Find the anti-commutation relations for the *Majorana* operators $j = 1, 2$

$$\gamma_j^A = \hat{c}_j + \hat{c}_j^\dagger \qquad \gamma_j^B = -i(\hat{c}_j - \hat{c}_j^\dagger).$$

Discuss if it is possible to define occupation numbers for *Majorana* fermions. [5]

(c) For $t = \mathit{\Delta} \neq 0$ show that the Hamiltonian given above can be written as

$$H = it\gamma_1^B \gamma_2^A.$$

Find the time dependence (in the Heisenberg picture) for the two *Majorana* operators that are not featured in this Hamiltonian, i.e. $\gamma_1^A(t)$ and $\gamma_2^B(t)$. [5]

(d) Verify the following operators have fermionic anti-commutation relations [4]

$$\hat{f}_1 = \tfrac{1}{2}\left(\gamma_1^A + i\gamma_2^B\right) \qquad \hat{f}_2 = \tfrac{1}{2}\left(\gamma_1^B + i\gamma_2^A\right).$$

(e) For $t = \mathit{\Delta} \neq 0$ rewrite the Hamiltonian in (c) in terms of the operators $\hat{f}_1$ and $\hat{f}_2$. Determine the eigenvalues of the Hamiltonian and their degeneracy. [6]

2 Consider scattering from a so-called *delta-shell potential* $V(\mathbf{r}) = V_0\delta(r - R)$.

(a) Write down the asymptotic solution to the 3D scattering problem. What is the scattering amplitude, the differential and total scattering cross section? In what sense is this an asymptotic solution rather than an exact solution? [5]

(b) Calculate the scattering amplitude for the delta-shell potential as given above in the first Born approximation. [5]

(c) Starting from the differential equation with boundary condition $u_0(0) = 0$

$$u_0''(r) + k^2 u_0(r) = \frac{2m}{\hbar^2} V(r) u_0(r),$$

find the phase shift $\delta_0(k)$ for s-wave scattering as a function of wave vector k

$$k \cot(kR + \delta_0) - k \cot kR = \frac{2mV_0}{\hbar^2}$$

for a particle with mass m. [5]

(d) Determine from the result in (c) the scattering length a_0 as a function of V_0. Sketch the function $a_0(V_0)$, choosing the appropriate dimensionless units, both in the repulsive and attractive case, and discuss its prominent features. [5]

(e) Compare the scattering amplitude obtained in the Born approximation in (b) and in partial wave analysis in (c) and (d), which are given also here,

$$f_{\text{Born}} = -\frac{2mV_0 R \sin qR}{\hbar^2 q} \qquad\qquad f_{s\text{-wave}} = -\frac{\frac{2mV_0R}{\hbar^2} R}{1 + \frac{2mV_0R}{\hbar^2}}.$$

In which aspects do they differ? In which limit do you expect them to agree? Show that they indeed lead to the same expression in that limit. [5]

3

(a) Define the density matrix ρ. Show that ρ is hermitian and that $\mathrm{tr}[\rho] = 1$. State the condition on $\mathrm{tr}[\rho^2]$ for pure and mixed states. [5]

(b) Consider the following Hamiltonian

$$H = -t\left(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1\right)$$

describing spinless fermions that can move between two sites, where $\hat{c}_1^\dagger$, $\hat{c}_1$, $\hat{c}_2^\dagger$, and $\hat{c}_2$ are the creation and annihilation operators for the sites 1 and 2.

For $N = \hat{c}_1^\dagger \hat{c}_1 + \hat{c}_2^\dagger \hat{c}_2 = 1$ determine the density matrix describing this system in equilibrium with a thermal reservoir at temperature T. [5]

Obtain the $T = 0$ and $T \to \infty$ limits and check the properties stated in part (a). [3]

Obtain the von Neumann entropy of the system at any temperature and the low and high temperature limits. [2]

(c) Consider now that the one spinless fermion in the system above (C) is coupled to another similar fermion in another two-state system (called B) described by operators $b_1^\dagger$, b_1, $b_2^\dagger$, and b_2. If the state of the global system is

$$|\Psi\rangle = \left(\frac{1}{2} b_1^\dagger c_1^\dagger + \frac{1}{\sqrt{2}} b_1^\dagger c_2^\dagger + \frac{1}{2} b_2^\dagger c_2^\dagger\right)|0\rangle,$$

where $|0\rangle$ is the vacuum state, obtain the global density matrix. [5]

Compute the reduced density matrix for system C, and indicate how you would compute its entanglement entropy. (This calculation is *not* required.) [5]

4 Given a harmonic oscillator with associated Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2,$$

where m is the mass of the oscillator and ω its angular frequency, and where x and p are the position and linear-momentum operators.

(a) State the equations of motion for the operators $x(t)$ and $p(t)$ in the Heisenberg picture and write down their general solution. Define the ladder operators, a and $a^\dagger$ (annihilation and creation operators), and obtain their time dependence in the Heisenberg picture. [5]

(b) Consider that the Hamiltonian is suddenly modified at $t = 0$ to

$$H = H_0 + \gamma x$$

i.e. displacing both the rest position of the oscillator and the minimum value of the harmonic potential. If the oscillator was in its ground state $|0\rangle$ prior to the change, compute the expectation value of the energy after the change with respect to the new potential energy minimum. Stating the expectation values of the position (with respect to the new minimum) and momentum operators just after the change. Using the solutions for $x(t)$ and $p(t)$ of part (a), write down $\langle x(t)\rangle$ and $\langle p(t)\rangle$. [5]

(c) Using the fact that the Hamiltonian eigenstates after the displacement relate to the old ones through a translation operator, $|n'\rangle = e^{-ipx_0/\hbar}|n\rangle$, and using the following relation,

$$e^A e^B = e^{A+B} e^{[A,B]/2} ,$$

derive the following expansion of the initial state $|\Psi(0)\rangle = |0\rangle$ in terms of new $|n'\rangle$ states just after the displacement [10]

$$|\Psi\rangle = \sum_{n'} \frac{1}{\sqrt{n!}} \left(\frac{m\omega x_0^2}{2\hbar} \right)^{n/2} e^{-\frac{m\omega x_0^2}{4\hbar}} |n'\rangle.$$

(d) Show that $|\Psi\rangle$ is an eigenstate of the annihilation operator for the displaced oscillator,

$$a'|\Psi\rangle = \alpha|\Psi\rangle$$

and that it is so for any time $t > 0$. State the eigenvalue α. [5]

END OF PAPER