

24 April 2019, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer **all four** questions.

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains five sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

- 1 (a) For a spin in a magnetic field that is described by the Hamiltonian $H = -\gamma \mathbf{B} \cdot \mathbf{S}$, find the Heisenberg equations of motion $\frac{d\mathbf{S}}{dt} = \frac{i}{\hbar}[H, \mathbf{S}]$ that $\mathbf{S}(t) = e^{iHt/\hbar} \mathbf{S} e^{-iHt/\hbar}$ obeys.

$$[\text{You may find useful the commutation relations } [S_j, S_k] = i\hbar \varepsilon_{jkl} S_l.] \quad [6]$$

- (b) For a spin-1/2 the following expression holds

$$\exp(-i\boldsymbol{\theta} \cdot \mathbf{S}/\hbar) = \mathbf{1} \cos(\theta/2) - i \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \sin(\theta/2) ,$$

where $\hat{\mathbf{n}}$ is a unit vector, $\boldsymbol{\theta} = \theta \hat{\mathbf{n}}$, and $\mathbf{S} = \frac{\hbar}{2} \boldsymbol{\sigma} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$, the latter being the Pauli matrices. Demonstrate its validity for $\hat{\mathbf{n}} = (1, 0, 0)$. [4]

- (c) Consider an electron in a hydrogen atom under the influence of an homogeneous, pulsed electric field along the x -direction. The system is described by the Hamiltonian,

$$H(t) = H_0 + \mathcal{E} \hat{x} \tau \sum_{n=0}^{\infty} \delta(t - n\tau) ,$$

where H_0 is the Hamiltonian for the electron in the unperturbed hydrogen atom, \mathcal{E} is the electric field magnitude, and τ is the time between pulses.

Limiting the Hilbert space to the one spanned by just two hydrogenic orbitals, $|1s\rangle = c_s^\dagger |0\rangle$ and $|2p_x\rangle = c_x^\dagger |0\rangle$ (where $|0\rangle$ is the vacuum state), the Hamiltonian can be written as

$$H(t) = \Delta(c_x^\dagger c_x - c_s^\dagger c_s) + \mathcal{E} x_0 (c_x^\dagger c_s + c_s^\dagger c_x) \tau \sum_{n=0}^{\infty} \delta(t - n\tau) ,$$

where $\Delta = \langle 2p_x | H_0 | 2p_x \rangle = -\langle 1s | H_0 | 1s \rangle$ and $x_0 = \langle 2p_x | \hat{x} | 1s \rangle = \langle 1s | \hat{x} | 2p_x \rangle$.

Show that the evolution operator from just before the first pulse to just before the $(N + 1)^{\text{th}}$, is given by

$$U(N\tau - \epsilon, -\epsilon) = (U_\Delta U_\mathcal{E})^N ,$$

where

$$U_\Delta = \begin{pmatrix} e^{i\tau\Delta/\hbar} & 0 \\ 0 & e^{-i\tau\Delta/\hbar} \end{pmatrix}, \quad U_\mathcal{E} = \begin{pmatrix} \cos \Phi_\mathcal{E} & -i \sin \Phi_\mathcal{E} \\ -i \sin \Phi_\mathcal{E} & \cos \Phi_\mathcal{E} \end{pmatrix}$$

and $\Phi_\mathcal{E} = \mathcal{E} x_0 \tau / \hbar$. [7]

- (d) Apply the expression in (b) to the evolution operator in (c) for $N = 1$ to find θ and \mathbf{n} . Interpret this result in terms of position and momentum of the electron before the second pulse.

$$[\text{The matrix of the momentum operator in this basis is proportional to } \sigma_y.]$$

[8]

- 2 A particle of energy $E = \frac{\hbar^2 k^2}{2m}$ is scattered from a localised potential $V(\mathbf{r})$.
 (a) Outline the essential elements of partial-wave analysis, leading to

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin \delta_l e^{i\delta_l} P_l(\cos \theta) \quad (\dagger)$$

defining all quantities that appear in this expression. [7]

- (b) Use (\dagger) to derive the optical theorem for the total scattering cross-section

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \text{Im} f(\theta = 0).$$

You may find it useful to know that $\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{2\delta_{ll'}}{2l+1}$ and $P_l(1) = 1$. [6]

- (c) Find the phase shift $\delta_0(k)$ for the s -wave scattering as a function of the wave vector k for a hard-sphere potential $V(r) = \infty$ for $r < R$, $V(r) = 0$ for $r > R$. Determine the s -wave scattering length a_0 and the s -wave scattering amplitude f_0 . Sketch the s -wave scattering cross-section $\sigma_0(k)$, choosing the appropriate dimensionless units, and discuss its (three) prominent features. [6]

- (d) Find the scattering amplitude for delta-shell potential $V(\mathbf{r}) = V_0 \delta(r - R)$ in the first Born approximation f_{Born} . Discuss the dependence of the scattering cross-section on the angle θ for small and large wave vector k . Does the first Born approximation you obtained in this part satisfy the optical theorem? [6]

- 3 (a) State the algebraic relations that creation and annihilation operators satisfy for bosons (fermions). Write down the number operators for bosons (fermions) and their possible eigenvalues for bosons (fermions). [5]

Consider the following Hamiltonian describing two interacting spins 1/2

$$H = -J(\hat{\sigma}_1^x \hat{\sigma}_2^x + \hat{\sigma}_1^y \hat{\sigma}_2^y) - h(\hat{\sigma}_1^z + \hat{\sigma}_2^z) \quad (\star)$$

where $\hat{\sigma}_j^\alpha$ with $\alpha = x, y, z$ and $j = 1, 2$ are the standard Pauli matrices.

- (b) Given the raising ($\hat{\sigma}_j^+$) and lowering ($\hat{\sigma}_j^-$) operators, $\hat{\sigma}_j^\pm = \hat{\sigma}_j^x \pm i\hat{\sigma}_j^y$, find the eigenvalues and eigenvectors of the Hamiltonian (\star). [4]

In one dimension one can rewrite spins in terms of fermionic operators

$$\hat{\sigma}_1^- = \hat{f}_1, \quad \hat{\sigma}_2^- = e^{i\pi \hat{f}_1^\dagger \hat{f}_1} \hat{f}_2, \quad \hat{\sigma}_j^z = 2\hat{f}_j^\dagger \hat{f}_j - 1$$

where \hat{f}_j with $j = 1, 2$ are the standard fermionic annihilation operators.

- (c) Use algebraic relations for fermionic operators to obtain the commutators

$$[\hat{\sigma}_1^-, \hat{\sigma}_1^+], \quad [\hat{\sigma}_1^z, \hat{\sigma}_2^z], \quad [\hat{\sigma}_1^-, \hat{\sigma}_2^-], \quad [\hat{\sigma}_2^-, \hat{\sigma}_2^+],$$

and show that they are consistent with the spin angular momentum algebra. [8]

- (d) Now rewrite the Hamiltonian (\star) in terms of fermionic operators and find the eigenvalues and eigenstates by diagonalising this bilinear Hamiltonian. [8]

4

(a) Define the density operator $\hat{\rho}$ of a quantum system that has probability p_i to be in each of the normalised states $|\psi_i\rangle$. Explain the distinction between mixed and pure states of a quantum system, and how this distinction is reflected in the density matrix. [4]

(b) Experimentalist A prepares a stream of atoms such that each atom is in state $|\psi_A\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$. Experimentalist B prepares a stream of atoms which is a mixture: half the atoms are in state $|\uparrow\rangle$ and half are in state $|\downarrow\rangle$. In the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ determine the density matrices ρ_A and ρ_B . [6]

(c) Show that streams A and B cannot be distinguished by making a measurement of σ_z , but that they can be told apart by measuring σ_x . [3]

(d) Experimentalist C prepares a stream which is an equal mixture of atoms in the two states $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. Can this be distinguished from streams A and B, and, if so, how? [3]

(e) Experimentalist D produces two streams of atoms flying in opposite directions, in such away that each atom flying to the left is entangled with another flying to the right, both forming a singlet of wavefunction

$$|\psi_D\rangle = \frac{1}{\sqrt{2}}(|\uparrow_L\downarrow_R\rangle - |\downarrow_L\uparrow_R\rangle),$$

in an experiment analogous to the one proposed by Einstein, Podolsky and Rosen in 1935. In the basis $\{|\uparrow_L\uparrow_R\rangle, |\uparrow_L\downarrow_R\rangle, |\downarrow_L\uparrow_R\rangle, |\downarrow_L\downarrow_R\rangle\}$: (i) compute the density matrix and check it corresponds to a pure state. (ii) Extract from it the reduced density matrix for measurements on the right stream, and (iii) calculate its entanglement entropy. [6]

(f) A stream prepared by experimentalist E is thermally equilibrated in a magnetic field $\mathbf{B} = (0, 0, B_z)$ at temperature T . The atoms have magnetic moment μ . Determine the density matrix ρ_E . [3]

END OF PAPER