NATURAL SCIENCES TRIPOS Part II

Wednesday 22 April 2020, 10.30 to 12.30

THEORETICAL PHYSICS 2

*Answer **all four** questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof. (The booklet is available for separate download.)

Please write answers in a manner that will be suitable for scanning and uploading. You will have 30 minutes after the end of the examination to scan and upload the answers. (See coversheet for further details.)

V7.1

A

1 A particle of mass M and energy $E = \hbar^2 k^2/2M$ scatters elastically from a spherically symmetric potential $V(r)$ which is zero for $r > R_0$.

(a) State what is meant by partial wave analysis. Explain in general terms (without mathematical detail) why scattering can be encoded in terms of phase shifts. [6]

(b) Writing the wavefunction for a particle with angular momentum quantum numbers ℓ and m_ℓ as

$$\Psi(\boldsymbol{r}) = Y_{\ell,m_\ell}(\theta,\phi)\frac{u_\ell(r)}{r}$$

where (r,θ,ϕ) are spherical polar co-ordinates, and $Y_{\ell,m_\ell}(\theta,\phi)$ is the spherical harmonic, show that the function $u_\ell(r)$ satisfies [5]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{r^2} + \frac{2M}{\hbar^2}V(r)\right]u_\ell(r) = k^2 u_\ell(r)\,.$$

[You may use $\nabla^2 = \frac{1}{r^2}\partial_r\left(r^2\frac{\partial}{\partial r}\right) - \frac{\boldsymbol{L}^2}{\hbar^2 r^2}$ where $\boldsymbol{L}^2$ is the squared total angular momentum operator.]

(c) Show that for $\ell = 0$ the function $u_0(r)$ at large radii, $r > R_0$, can be written as

$$u_0(r) \propto \sin(kr + \delta_0) \qquad (\star)$$

and explain the physical meaning of the phase shift δ_0. [5]

A certain potential has a bound state for $\ell = 0$ at an energy $E = -E_\mathrm{B}$ with E_B positive.

(d) Show that for $r > R_0$ the bound state wavefunction is of the form

$$u_0^\mathrm{B}(r) \propto e^{-\kappa r}$$

and determine the value of κ. Hence show that, for $E_\mathrm{B} \ll \hbar^2/MR_0^2$ the bound state wavefunction is mostly at $r > R_0$. [4]

(e) Assuming $E_\mathrm{B} \ll \hbar^2/MR_0^2$, use orthogonality of the scattering state $(\star)$ with $u_0^\mathrm{B}(r)$, or otherwise, to determine the phase shift δ_0 for low-energy scattering. [5]

[Hint: In this limit, you can compute the overlap by considering $R_0 \to 0$ and integrating the functions over $r = 0 \to \infty$.]

2 (a) Write the density matrix in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis for an ensemble of spin-1/2 particles in each of the following situations: [6]

(i) each particle is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$;

(ii) an equal statistical mixture of states $|\uparrow\rangle$ and $|\downarrow\rangle$;

(iii) an equal statistical mixture of states $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$.

(b) Which of the above situations are physically distinct? What observable can be measured to differentiate them? Find the expectation of this observable from the three density matrices. [5]

[*The spin operators are described by Pauli matrices in the* $\{|\uparrow\rangle, |\downarrow\rangle\}$ *basis:*

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$
]

Two spin-1/2 particles interact via the Hamiltonian

$$H = -2J(\mathbf{S}_1 \cdot \mathbf{S}_2/\hbar^2 + 1/4) .$$

Writing the time-evolution operator $U(t) = e^{-iHt/\hbar}$ in the basis of states $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ gives the matrix

$$U(t) = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & \cos\theta & i\sin\theta & 0 \\ 0 & i\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix} ,$$

where $\theta \equiv Jt/\hbar$.

(c) Suppose we prepare the system in the state $|\uparrow\downarrow\rangle$ at $t = 0$. Find the density matrix at time t using $\rho(t) = U(t)\rho(0)U^\dagger(t)$. What pure state does it describe? [5]

(d) Find the reduced density matrix $\rho_{\text{red}}(t)$ describing the first spin by tracing over the latter. [4]

(e) Find the entanglement entropy $S_{\text{ent}}(t)$ from the eigenvalues of ρ_{red}. When is the entanglement minimised, and maximised? What are the corresponding states of the total system? [5]

3 (a) Consider a particle in a time-dependent harmonic well given by the Hamiltonian

$$H(t) = \frac{p^2}{2M} + \frac{1}{2}M(\omega(t))^2x^2 \ .$$

Write the instantaneous Hamiltonian $H(t)$ in terms of the ladder operators $a(t)$ and $a^\dagger(t)$, where

$$a(t) = \frac{1}{\sqrt{2}}\left(\frac{x}{\xi(t)} + i\frac{\xi(t)}{\hbar}p\right) \quad \text{and} \quad \xi(t) \equiv \sqrt{\frac{\hbar}{M\omega(t)}} \ , \qquad (*)$$

and state its eigenvalues. (You may directly quote the results.) [4]

(b) The particle is initially in the ground state $|0\rangle$ of $H(t=0)$. The frequency $\omega(t)$ is ramped from ω_0 to $2\omega_0$ over an interval τ, i.e., $\omega(t) = \omega_0(1+t/\tau)$. Without any calculation, argue what condition τ and ω_0 must satisfy for the ramp to be "slow" or *adiabatic*. What would be the final energy in this case? [4]

(c) By expressing x in terms of a_0 and $a_0^\dagger$, show that $H(t)$ can be written as

$$H(t) = H_0 + \frac{1}{4}\hbar\omega_0\left[\left(\frac{\omega(t)}{\omega_0}\right)^2 - 1\right]\left(a_0 + a_0^\dagger\right)^2 \ ,$$

where H_0 is the Hamiltonian for $t=0$ and $a_0 \equiv a(0)$. $\left[\textit{Use } (*) \text{ at } t=0.\right]$ [4]

(d) Use this expression to find the final energy expectation $\langle H(\tau)\rangle$ for a sudden ramp, where the particle has no time react. Why is this energy different from that in the adiabatic case? $\left[\textit{Note: } a_0|0\rangle = 0 \textit{ and } [a(t), a^\dagger(t)] = 1.\right]$ [5]

(e) At any point during the ramp, the state $|\psi(t)\rangle$ can be written as a superposition of the instantaneous energy eigenstates $|n\rangle_t$,

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle_t \ ,$$

where the states are ordered in increasing energy. Assuming we start from the ground state at $t=0$, which of the coefficients $c_n(t)$ will always be zero? Use symmetry arguments (no calculation required). [3]

(f) Use the time-dependent Schrödinger equation and the result

$$\frac{d|n\rangle_t}{dt} = \frac{\hbar\omega_0}{2\tau}\sum_{m\neq n}\frac{{}_t\langle m|\left[a(t)+a^\dagger(t)\right]^2|n\rangle_t}{E_n(t)-E_m(t)}|m\rangle_t$$

to write the equation of motion for the coefficients in the form

$$\frac{dc_m(t)}{dt} = A_m(t)c_m(t) + \sum_{n\neq m} B_{mn}(t)c_n(t) \ .$$

You do not need to evaluate inner products. Compare the frequency scales in A_m and B_{mn} to explain how the adiabaticity condition is recovered. [5]

4 The Hamiltonian for a set of interacting bosons is $\hat{H} = \hat{H}_{\rm kin} + \hat{H}_{\rm int}$ where

$$\hat{H}_{\rm kin} = -\frac{\hbar^2}{2M}\int d^3\boldsymbol{r}\ \hat{\psi}^\dagger(\boldsymbol{r})\boldsymbol{\nabla}^2\hat{\psi}(\boldsymbol{r})\,,$$

$$\hat{H}_{\rm int} = \frac{1}{2}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\ \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}')U(\boldsymbol{r}-\boldsymbol{r}')\hat{\psi}(\boldsymbol{r}')\hat{\psi}(\boldsymbol{r})\,.$$

and the bosonic field operators, $\hat{\psi}^\dagger(\boldsymbol{r})$ and $\hat{\psi}(\boldsymbol{r})$, satisfy $[\hat{\psi}(\boldsymbol{r}),\hat{\psi}^\dagger(\boldsymbol{r}')] = \delta^3(\boldsymbol{r}-\boldsymbol{r}')$, $[\hat{\psi}(\boldsymbol{r}),\hat{\psi}(\boldsymbol{r}')] = [\hat{\psi}^\dagger(\boldsymbol{r}),\hat{\psi}^\dagger(\boldsymbol{r}')] = 0\,.$

(a) The plane-waves $\phi_{\boldsymbol{k}}(\boldsymbol{r}) = \frac{1}{\sqrt{V}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ provide an orthonormal basis of single-particle states in a system of volume V. Show that the creation and annihilation operators for these states, denoted by $\hat{a}^\dagger_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}$, satisfy $[\hat{a}_{\boldsymbol{k}},\hat{a}^\dagger_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}$, $[\hat{a}_{\boldsymbol{k}},\hat{a}_{\boldsymbol{k}'}] = [\hat{a}^\dagger_{\boldsymbol{k}},\hat{a}^\dagger_{\boldsymbol{k}'}] = 0\,,$ and that [8]

$$\hat{H}_{\rm kin} = \sum_{\boldsymbol{k}}\frac{\hbar^2|\boldsymbol{k}|^2}{2M}\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}}\,.$$

(b) Show that the interaction energy $\hat{H}_{\rm int}$ can be written in terms of the density operator $\hat{\rho}(\boldsymbol{r}) \equiv \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})$, by combining both two-body and one-body terms. [4]

(c) Show that the Fourier transform of the density operator, $\hat{\rho}_{\boldsymbol{q}} \equiv \int d^3\boldsymbol{r}\hat{\rho}(\boldsymbol{r})e^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}$, may be written [3]

$$\hat{\rho}_{\boldsymbol{q}} = \sum_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}-\boldsymbol{q}}\hat{a}_{\boldsymbol{p}}\,.$$

Starting from the exact groundstate, $|{\rm GS}\rangle$, approximate descriptions of the low-energy excitations of $\hat{H}$ can be constructed by the variational states

$$|\Psi_{\boldsymbol{q}}\rangle = \frac{1}{\sqrt{S_{\boldsymbol{q}}}}\hat{\rho}_{\boldsymbol{q}}|{\rm GS}\rangle\,,$$

with $\hat{\rho}_{\boldsymbol{q}}$ the Fourier transform of the density operator and $S_{\boldsymbol{q}}$ a normalization factor. The variational energy can be shown to be

$$\Delta_{\boldsymbol{q}} = \frac{1}{2S_{\boldsymbol{q}}}\langle{\rm GS}|\left[\hat{\rho}^\dagger_{\boldsymbol{q}},\left[\hat{H},\hat{\rho}_{\boldsymbol{q}}\right]\right]|{\rm GS}\rangle\,.$$

(d) Show that $|\Psi_{\boldsymbol{q}}\rangle$ is orthogonal to $|{\rm GS}\rangle$ for $\boldsymbol{q}\neq 0$, provided $|{\rm GS}\rangle$ describes a state of uniform particle density. [2]

(e) Explain why $\left[\hat{H}_{\rm int},\hat{\rho}_{\boldsymbol{q}}\right] = 0$. Hence show that

$$\left[\hat{H},\hat{\rho}_{\boldsymbol{q}}\right] = -\frac{\hbar^2}{M}\sum_{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{q}\ \hat{a}^\dagger_{\boldsymbol{k}-\boldsymbol{q}/2}\hat{a}_{\boldsymbol{k}+\boldsymbol{q}/2}\,.$$

Deduce an expression for the variational energy $\Delta_{\boldsymbol{q}}$. [8]

[*You may use the following identity without proof:*
$[AB,CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$]

END OF PAPER