UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 28 April 2021, 2pm to 4pm

THEORETICAL PHYSICS 2

Answer ***all four*** *questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains **four** *sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof. (The booklet is available for separate download.)*

Please write answers in a manner that will be suitable for scanning and uploading. You will have 30 minutes after the end of the examination to scan and upload the answers.

V7.1

1 A set of laser fields couple two electronic levels in an atom at position $\boldsymbol{r} = (x, y)$, giving rise to a position-dependent effective potential

$$\mathbb{V}(\boldsymbol{r}) = V_0 \begin{pmatrix} \sinh y & \mathrm{e}^{\mathrm{i}kx} \\ \mathrm{e}^{-\mathrm{i}kx} & -\sinh y \end{pmatrix}$$

expressed here in the basis of the two electronic levels.

Consider first the situation in which the atom is moved at a constant speed v along the line $y = 0$, such that $\boldsymbol{r}(t) = (x, y) = (vt, 0)$.

(a) Determine the instantaneous energies $E_\pm(t)$ and instantaneous eigenstates $|\pm, \boldsymbol{r}(t)\rangle$ for a stationary atom at the position $\boldsymbol{r}(t) = (x, y) = (vt, 0)$. [5]

(b) By expanding the wavefunction as

$$|\Psi(t)\rangle = c_+(t)|+, \boldsymbol{r}(t)\rangle + c_-(t)|-, \boldsymbol{r}(t)\rangle$$

and using the time-dependent Schrödinger equation, show that the electronic state of the atom will evolve adiabatically for $\hbar|\langle -, \boldsymbol{r}(t)|\frac{d}{dt}|+, \boldsymbol{r}(t)\rangle \ll V_0$. Use your results from (a) to deduce the condition on velocity v for which adiabaticity holds. [8]

Now consider the atom to have a mass m, and to move freely in the x, y-plane under the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 \otimes \mathbb{1} + \mathbb{V}(\boldsymbol{r})$$

where $\mathbb{1}$ is the identity operator in the space of the two electronic levels and $\mathbb{V}(\boldsymbol{r})$ is the optical potential described above, whose local eigenstates are $|\pm, \boldsymbol{r}\rangle$.

By expanding the wavefunction as

$$|\Psi(\boldsymbol{r})\rangle = \psi_+(\boldsymbol{r})|+, \boldsymbol{r}\rangle + \psi_-(\boldsymbol{r})|-, \boldsymbol{r}\rangle$$

and *making the adiabatic assumption*, the time-independent Schrödinger equation, $H|\Psi(\boldsymbol{r})\rangle = E|\Psi(\boldsymbol{r})\rangle$ can be written as the two equations

$$\langle \pm, \boldsymbol{r}| \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \mathbb{V}(\boldsymbol{r})\right] \psi_\pm(\boldsymbol{r})|\pm, \boldsymbol{r}\rangle = E\psi_\pm(\boldsymbol{r}) \qquad (\star)$$

for the wavefunctions $\psi_\pm(\boldsymbol{r})$ describing atoms in the two energy levels.

(c) Show that the equations $(\star)$ lead to effective Hamiltonians $H_\pm\psi_\pm(\boldsymbol{r}) = E\psi_\pm(\boldsymbol{r})$, with

$$H_\pm = \frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\boldsymbol{\nabla} + \hbar\boldsymbol{A}_\pm\right)^2 + V_\pm(\boldsymbol{r})$$

where $\boldsymbol{A}_\pm(\boldsymbol{r}) = -\mathrm{i}\langle \pm, \boldsymbol{r}|\boldsymbol{\nabla}|\pm, \boldsymbol{r}\rangle$. Deduce a general expression for $V_\pm(\boldsymbol{r})$. [8]

(d) Outline the conditions under which the adiabaticity assumption leading to $(\star)$ holds. [4]

2 The propagator for a quantum particle moving in one dimension is defined as

$$K(x,t|x',t') \equiv \theta(t-t')\langle x|U(t,t')|x'\rangle$$

where $|x\rangle$ are the position eigenstates, $U(t,t')$ is the time-evolution operator from initial time t' to final time t, and $\theta(t)$ is the Heaviside step function.

(a) Show that, for a Hamiltonian $H(t) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x,t)$, the propagator satisfies

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t} - H(t)\right] K(x,t|x',t') = \mathrm{i}\hbar\delta(t-t')\delta(x-x')$$

with the boundary condition that $K(x,t|x',t') = 0$ for $t < t'$. [5]

For a simple harmonic oscillator, with Hamiltonian $H_0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2x^2$, the propagator starting from $x' = 0$ at $t' = 0$ is

$$K_0(x,t|0,0) = \theta(t)\sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega t}}\exp\left(\mathrm{i}\frac{m\omega}{2\hbar}x^2\cot\omega t\right) .$$

(b) Show that, in the presence of a time-dependent force, $H = H_0 - xF(t)$, the solution to the equation in part (a) can be found, for $x' = t' = 0$, by writing

$$K(x,t|0,0) = K_0(x,t|0,0)\exp\left[\mathrm{i}G(x,t)/\hbar\right]$$

with $G(x,t) = A(t) + B(t)x$, provided that

$$\begin{aligned} \dot{A}(t) &= -\frac{B(t)^2}{2m} \\ \dot{B}(t) + \omega\cot(\omega t)B(t) &= F(t). \end{aligned}$$

Hence, find an expression for $B(t)$. [12]
[*Hint: multiply the differential equation by* $\sin\omega t$.]

(c) By making reference to the Feynman path integral formulation of quantum mechanics, comment on the relation of these results to the classical trajectory. [8]

3 A system of N bosons is described by the second-quantized Hamiltonian

$$H = \int \mathrm{d}\boldsymbol{r}\ \psi^\dagger(\boldsymbol{r})\Big[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_1(\boldsymbol{r})\Big]\psi(\boldsymbol{r}) + \frac{1}{2}\iint \mathrm{d}\boldsymbol{r}\ \mathrm{d}\boldsymbol{r}'\psi^\dagger(\boldsymbol{r})\,\psi^\dagger(\boldsymbol{r}')V_2(\boldsymbol{r},\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r})$$

where $\psi(\boldsymbol{r})$ and $\psi^\dagger(\boldsymbol{r})$ are the field operators, and $V_2(\boldsymbol{r},\boldsymbol{r}')$ is the two-body interaction potential.

(a) State the set of commutation relations that the field operators satisfy. [3]

(b) Hence show that

$$\left[\psi(\boldsymbol{r}), \int \mathrm{d}\boldsymbol{r}' \left(\frac{\partial}{\partial \boldsymbol{r}'}\psi^\dagger(\boldsymbol{r}')\right) F(\boldsymbol{r}')\right] = -\boldsymbol{\nabla} F(\boldsymbol{r})$$

where $F(\boldsymbol{r})$ commutes with $\psi(\boldsymbol{r})$ and vanishes at large $|\boldsymbol{r}|$. [2]

(c) Write down a second-quantized form for the total momentum operator, $\boldsymbol{P}$. By employing appropriate commutation relations, and assuming that $\psi(\boldsymbol{r})$ and $\psi^\dagger(\boldsymbol{r})$ vanish at large $|\boldsymbol{r}|$, show that the total momentum $\boldsymbol{P}$ is conserved when $V_1 = 0$ and $V_2(\boldsymbol{r},\boldsymbol{r}') = V_2(|\boldsymbol{r}-\boldsymbol{r}'|)$. [8]

[*You may use the facts that* $[AB,C] = A[B,C] + [A,C]B$; $[A,BC] = B[A,C] + [A,B]C$; *and* $\mathrm{d}|\boldsymbol{r}|/\mathrm{d}\boldsymbol{r} = \boldsymbol{r}/|\boldsymbol{r}|$.]

(d) Defining the current density operator as

$$\boldsymbol{J}(\boldsymbol{r}) = \frac{\hbar}{2mi}\Big\{\psi^\dagger(\boldsymbol{r})\boldsymbol{\nabla}\psi(\boldsymbol{r}) - [\boldsymbol{\nabla}\psi^\dagger(\boldsymbol{r})]\psi(\boldsymbol{r})\Big\}\,,$$

show that the particle density $\rho(\boldsymbol{r}) = \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0\,,$$

in the Heisenberg picture. [8]

(e) How would each of the above answers be affected if the particles were fermions instead of bosons? [4]

4 The equation of motion for the density operator of a quantum system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \frac{\mathrm{i}}{\hbar}\left[\rho(t), H\right] + \gamma\mathcal{L}[\rho(t)]$$

where H is the Hamiltonian and

$$\mathcal{L}[\rho] \equiv L\rho L^\dagger - \frac{1}{2}\left(L^\dagger L\rho + \rho L^\dagger L\right)$$

describes dissipative coupling to an external environment via some operator L.

(a) Show that the above equation of motion for $\rho(t)$ leaves $\mathrm{Tr}[\rho(t)]$ time-independent. [4]

$\big[$*You may use the fact that* $Tr[AB] = Tr[BA]$.$\big]$

(b) Show that for a two-level quantum system one can parameterise $\rho(t)$ in terms of three quantities $\rho_{i=x,y,z}(t)$ as

$$\rho(t) = \frac{1}{2}\mathbb{1} + \sum_{i=x,y,z} \rho_i(t)\sigma_i\,,$$

where $\mathbb{1}$ is the 2×2 identity matrix and $\sigma_{x,y,z}$ are the Pauli matrices. [4]

$\left[\right.$ *The Pauli matrices are* $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\left.\right]$

Consider a two-level system, with Hamiltonian $H = \hbar\Delta\sigma_z$ and dissipator $L = \sigma_-$, where $\sigma_\pm \equiv \sigma_x \pm \mathrm{i}\sigma_y$.

(c) Show that the equation of motion for $\rho_z(t)$ is [8]

$$\dot{\rho}_z = -4\gamma\rho_z - 2\gamma\,.$$

$\big[$*You may use:* $[\sigma_l, \sigma_m] = 2\mathrm{i}\epsilon_{lmn}\sigma_n$, $\sigma_+\sigma_- = 2(\mathbb{1} + \sigma_z), \sigma_-\sigma_+ = 2(\mathbb{1} - \sigma_z)$, *and* $\sigma_-\sigma_x\sigma_+ = \sigma_-\sigma_y\sigma_+ = 0$, $\sigma_-\sigma_z\sigma_+ = 2(\mathbb{1} - \sigma_z)$.$\big]$

(d) Given that the remaining equations of motion are $\dot{\rho}_x = -2\Delta\rho_y - 2\gamma\rho_x$ and $\dot{\rho}_y = 2\Delta\rho_x - 2\gamma\rho_y$, determine the steady state density operator to which the system evolves at long times, $t \to \infty$. How is it related to the eigenstates of the Hamiltonian? [4]

The equation of motion is extended to include a second dissipator, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \frac{\mathrm{i}}{\hbar}\left[\rho, H\right] + \gamma\mathcal{L}[\rho] + \gamma'\mathcal{L}'[\rho]$$

with $\mathcal{L}'[\rho]$ of the same form as $\mathcal{L}[\rho]$ but with $L = \sigma_-$ replaced by $L' = \sigma_+ \equiv \sigma_x + \mathrm{i}\sigma_-$.

(e) For what value of γ'/γ does the steady state density operator, at $t \to \infty$, describe thermal equilibrium at temperature T? [5]

$\big[$*You may use:* $\sigma_-\sigma_x\sigma_+ = \sigma_+\sigma_y\sigma_- = 0$, $\sigma_+\sigma_z\sigma_- = -2(\mathbb{1} + \sigma_z)$.$\big]$