UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 27 April 2022, 10.30 to 12.30

THEORETICAL PHYSICS 2

*Answer **all four** questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains **five** *sides, including this one, and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof. (The booklet is available for separate download.)*

V7.1

1 Consider the following Hamiltonian for a spin-1 degree of freedom in a magnetic field:

$$H = \mathbf{B} \cdot \mathbf{S},$$

where $\mathbf{S} = (S_x, S_y, S_z)$ and $\mathbf{B} = B_0(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

(a) Write the explicit Hamiltonian in terms of the *spin-1* matrices $S_{x,y,z}$ in the $|m_z\rangle$-basis, where $m_z = -1, 0, 1$ for $s = 1$.
$\big[$*Hint: Recall that for spin-s, the raising and lowering operators act as* $S_{\pm}|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}|s, m_s\rangle$. *Note that* $|m_z\rangle \equiv |s = 1, m_s\rangle$.$\big]$ [2]

(b) Use a 'rotated frame' transformation $U(\theta, \phi) = e^{i\phi S_z} e^{i\theta S_y} \equiv U(\phi)U(\theta)$ to show that

$$H = U(\theta, \phi) H_z (U(\theta, \phi))^{\dagger},$$

in terms of $H_z = B_0 S_z$.
$\big[$*Hint: Recall that* $e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ...\big]$ [8]

The eigenstates of H are thus given by $|B, m_z\rangle = U(\theta, \phi)|m_z\rangle$.

(c) Show that the explicit form of the eigenstate with highest eigenvalue, $|B, +1\rangle = U(\theta, \phi)|m_z = 1\rangle$, is given by [7]

$$|B, +1\rangle = \begin{pmatrix} e^{-i\phi}\cos^2(\theta/2) \\ \sqrt{2}\sin(\theta/2)\cos(\theta/2) \\ e^{i\phi}\sin^2(\theta/2) \end{pmatrix}.$$

(d) Show that the Berry potential $\mathbf{A}_+ = -i\langle B, +1|\nabla|B, +1\rangle$ is [5]

$$-\frac{1}{B_0}\cot(\theta)\,\hat{\phi}.$$

(e) Writing $\mathbf{B} = B_0\hat{\mathbf{n}}$, show that the Berry curvature is given by $\frac{\hat{\mathbf{n}}}{B_0^2}$. What does this expression physically mean? Predict a general relation for a spin-s degree of freedom in a magnetic field with the same Hamiltonian as above. [3]

2 Consider a one-dimensional potential of strength $V_0 > 0$ of the form,

$$V(x) = \begin{cases} V_0, & |x| < a\,, \\ 0, & |x| \geq a\,. \end{cases}$$

A solution of the scattering problem with energy $E = \frac{\hbar k^2}{2m}$ is a solution of the integral equation

$$\Psi_k(x) = e^{ikx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_k^+(x, x') V(x') \Psi_k(x').$$

(a) Show that the retarded Green's function for this system is $G_k^+(x, x') = -\frac{im}{\hbar^2 k} e^{ik|x-x'|}$. $G_k^+(x, x')$ can also be written as the position matrix elements of an operator $\hat{O}$. Give an expression of $\hat{O}$. [11]

(b) By writing the wave function far away from the scatterer ($|x| \to \infty$) as

$$\Psi_k(x) \approx e^{ikx} + e^{ik|x|} f(k, k'),$$

for $k' = k\,\mathrm{sgn}(x)$, find an expression for the scattering amplitude $f(k, k')$ in terms of V_0 and a. [5]

(c) For waves incident from the left ($x \to -\infty$), find the incident, reflected and transmitted waves and their respective probability currents j_i, j_r and j_t in terms of $f(k, k')$. Determine the transmission, $T = j_t/j_i$, and reflection, $R = j_r/j_i$, coefficients. Show that the scattering amplitude must satisfy

$$\mathrm{Re}\{f(k, k)\} = -\frac{1}{2}\{|f(k, k)|^2 + |f(k, -k)|^2\}$$

for the probability to be conserved. [5]

(d) As a limiting case, consider a delta function potential $V(x) = g\delta(x)$, of strength g, and determine the scattering amplitude $f(k)$. (Notice that in this case $f(k)$ depends on only one reciprocal wave vector.) [4]

3 The Helium atom has Hamiltonian $H = H_0 + H_{12}$, where

$$H_0 = \frac{1}{2m}\left[\mathbf{p}_1^2 + \mathbf{p}_2^2\right] - 2e^2\left[\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|}\right], \quad H_{12} = \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$

(a) We first address the spin part. Show that the two spins can form a triplet χ_{1,m_s} (with $m_s = -1, 0, 1$) or singlet $\chi_{0,0}$ ($m_s = 0$) wave function. Express them in the S_z-eigenstate basis, $|\uparrow\rangle$ and $|\downarrow\rangle$. What are their symmetry properties? [3]

(b) What is the symmetry property of fermionic states? Give the form of the ground state if H_{12} is ignored. You may use, without derivation, the fact that the Hydrogen problem has eigenfunctions $\psi_{nlm}(\mathbf{r})$ with eigenenergies $E_{nlm} \propto -e^4/(2\hbar n^2)$, for principal quantum number n, azimuthal quantum number $l = 0, 1, ..., n-1$ and magnetic quantum number $m = -l, ..., +l$.
[*Hint: Consider decomposing the wave function into a spatial part and spin part. The latter can be left implicit.*] [5]

(c) Determine the set of degenerate eigenstates of the next energy level using the same analysis as in (b). [6]

(d) Consider now the repulsion term H_{12} as a perturbation and give an expression for the correction to the ground state energy. [6]

[*Hint: Use here that,*

- *The wave function of the Hydrogen 1s state* $\psi_{1,0,0}(r)$ *reads* $1/\sqrt{a_0^3\pi}e^{-r/a_0}$. *Note that* a_0 *needs to be replaced with* $a_0/2$ *to account for the atomic nucleus in Helium.*
- $\int d\phi_1 \int d(\cos\theta_1) \int d\phi_2 \int d(\cos\theta_2) \frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} = 8\pi^2 \int d(\cos\theta) \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} = \frac{8\pi^2}{r_1r_2}(r_1 + r_2 + |r_1 - r_2|)$.
- $\int_0^\infty dx\, x\, e^{-4x} \int_0^\infty dy\, y\, e^{-4y}(x + y + |x - y|) = \frac{5}{2048}$.

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(e) The "classical" expression for density-density fluctuations,

$$H_{int} = \frac{1}{2}\int d\mathbf{r}\, d\mathbf{r}'\, \rho(\mathbf{r})U(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}'),$$

needs to be modified when ρ is expressed in second quantized form since the ordering for indistinguishable particles matters. Give the correct expression of the expectation value for both fermionic and bosonic product states in terms of the density operator $\rho(\mathbf{r})$ and the single particle density matrix $g(\mathbf{r}, \mathbf{r}')$, and comment on the result. [5]

4 Consider two spin-1/2 particles in the state

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 \right).$$

(a) Imagine that we perform a measurement of $S_{z,1}$ to determine the z-component of the spin of the first particle. Comment on whether the result of a simultaneous measurement of $S_{z,2}$ can always be predicted from the first measurement. [4]

(b) Determine the two-spin density matrix ρ corresponding to the state $|\Phi\rangle$ and compute the reduced density matrix for the first particle $\rho_{1,\text{red}} = \text{tr}_2[\rho]$ by tracing out the second spin. Then calculate the degree of entanglement, $S_{\text{ent}} = -\text{tr}[\rho_{1,\text{red}} \log \rho_{1,\text{red}}]$, between the two particles and comment on the purity of ρ and $\rho_{1,\text{red}}$. [6]

(c) Now, consider the state

$$|\Psi\rangle = \frac{1}{2} \left(|\uparrow\rangle_1 |\uparrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 + |\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\downarrow\rangle_2 \right),$$

and determine the reduced density matrix $\rho_{1,\text{red}}$ in this case. Find the degree of entanglement. How does your result compare with the one obtained in (b)? [5]

Now, consider a single spin-1/2 particle in the state $|S_z, +\rangle = |\uparrow\rangle$.

(d) Imagine that the particle is going through a Stern-Gerlach device oriented in the $x - z$ plane, $\hat{n} = -\sin\theta \hat{e}_x + \cos\theta \hat{e}_z$. Determine the probability of measuring the outgoing particle in the $|S_n, +\rangle$ state (the state with eigenvalue $+\frac{1}{2}$). [5]

(e) Consider a variable Stern-Gerlach device which can have one of three different orientations with equal probability,

$$\begin{aligned} \hat{n}_1 &= -\sin\theta \hat{e}_x + \cos\theta \hat{e}_z, \\ \hat{n}_2 &= -\sin(\theta - \frac{2\pi}{3})\hat{e}_x + \cos(\theta - \frac{2\pi}{3})\hat{e}_z, \\ \hat{n}_3 &= -\sin(\theta + \frac{2\pi}{3})\hat{e}_x + \cos(\theta + \frac{2\pi}{3})\hat{e}_z. \end{aligned}$$

For an incoming particle in state $|S_z, +\rangle$, determine the probability of measuring the eigenvalue $+\frac{1}{2}$ after the particle has traversed the device. [5]