UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 26 April 2023, 10.30 to 12.30

THEORETICAL PHYSICS 2

*Answer **all four** questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains **five** *sides, including this one, and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof. (The booklet is available for separate download.)*

V7.1

1 Consider a dimerized chain with two sites a, b in N unit cells described by the Hamiltonian

$$H = \sum_{j=1}^{N} (t + \delta t) c^{\dagger}_{a,j} c_{b,j} + (t - \delta t) c^{\dagger}_{a,j+1} c_{b,j} + h.c.,$$

where $h.c.$ refers to the Hermitian conjugate, and t and δt are hopping parameters that we leave implicit.

(a) Using a Fourier transform $c_{j,\alpha} = \frac{1}{\sqrt{N}} \sum_k c_{k,\alpha} e^{ikj}$, show that the above Hamiltonian can be rewritten as

$$H(k) = \sum_{k,\alpha,\beta} H_{\alpha,\beta}(k) c^{\dagger}_{\alpha,k} c_{\beta,k},$$

where $\alpha, \beta \in a, b$ and $H_{\alpha,\beta}(k) = \mathbf{d}(k) \cdot \boldsymbol{\sigma}$ in terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and vector $\mathbf{d}(k) = ((t + \delta t) + (t - \delta t)\cos(k), (t - \delta t)\sin(k), 0)$. [3]

(b) Show that the spectrum consists of two energy bands,

$$E(k) = \pm\sqrt{((t + \delta t) + (t - \delta t)\cos(k))^2 + ((t - \delta t)\sin(k))^2}.$$

[2]

(c) Assuming now that $\delta t = -t$, show that the eigenstates $|u_k, \pm\rangle$ read

$$|u_k, \pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm e^{-ik} \\ 1 \end{pmatrix}$$

and that evaluating the Berry potential $\mathbf{A}_{\pm} = -i\langle u_k, \pm|\nabla|u, \pm\rangle$ over the closed path $k \to k + 2\pi$ renders $\oint A_{\pm} \mathrm{d}k = -\pi$. (Note that $A_{\pm}$ is just a scalar in our one-dimensional case.) [5]

(d) Assuming that $\delta t = t$, show that the eigenstates $|u_k, \pm\rangle$ are

$$|u_k, \pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$$

and that evaluating the Berry potential $A_{\pm} = -i\langle u_k, \pm|\nabla|u, \pm\rangle$ over the closed path $k \to k + 2\pi$ renders $\oint A_{\pm} \mathrm{d}k = 0$. [5]

(e) Show that for general δt, the eigenstates are given as

$$|u_k, \pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm e^{i\phi_k} \\ 1 \end{pmatrix},$$

where $\phi_k = \tan^{-1} \frac{v \sin k}{v + w \cos k}$, for $v = (t + \delta t)$ and $w = (t - \delta t)$. Why is $\oint A_{\pm} \mathrm{d}k = 0$ over the closed path $k \to k + 2\pi$ undefined when $v = w$? What does it evaluate to for $v > w$ and $v < w$ and how does this compare to the above answers? Finally, how does $\oint A_{\pm} \mathrm{d}k = 0$ over the closed path $k \to k + 2\pi$ change when we assign a different gauge to the eigenstates $|u_k, \pm\rangle \to e^{i\varphi_k}|u_k, \pm\rangle$? [10]

Solution 1. (a) *[Unseen but similar to derivations addressed in Lecture.]*
Inserting the Fourier modes in the real space Hamiltonian and using that $\frac{1}{N}\sum_j e^{i(k-k')j} = \delta_{k,k'}$, we directly verify the result. Here we not for the second disperse term that the 'unmatched' e^{ik} render the sine and cosine terms.

(b) *[Unseen but similar to derivations addressed in Lecture.]*
As $H_{\alpha,\beta}(k) = \mathbf{d}(k)\cdot\boldsymbol{\sigma}$, the energies are easily found to be $E_\pm = \pm|\mathbf{d}(k)| = \pm\sqrt{d_x^2 + d_y^2 + d_z^2}$.

(c) *[Unseen but similar to derivations addressed in Lecture.]*
For $t = -\delta t$, we obtain

$$H_{\alpha,\beta} = \begin{pmatrix} 0 & 2te^{-ik} \\ 2te^{ik} & 0 \end{pmatrix}. \tag{1}$$

Hence, we directly verify that

$$|u_k, \pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm e^{-ik} \\ 1 \end{pmatrix}$$

are solutions with eigenvalues $\pm 2t$ consistent with (b). The Berry potential simply evaluates to $-\frac{1}{2}$ and hence evaluating this over the closed path [the cycle of the Brillouin zone] results in $-\pi$.

(d) *[Unseen but similar to derivations addressed in Lecture.]*
For $t = \delta t$, we obtain

$$H_{\alpha,\beta} = \begin{pmatrix} 0 & 2t \\ 2t & 0 \end{pmatrix}. \tag{2}$$

Hence, we directly verify that

$$|u_k, \pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$$

are solutions with eigenvalues $\pm 2t$ consistent with (b). The Berry potential simply evaluates to $-\frac{1}{2}$ and hence evaluating this over the closed path [the cycle of the Brillouin zone] results in $-\pi$. This makes sense as for changing $\delta t = -t$ to $\delta t = t$ charges are flipped over half a unit cell giving the result as the Berry, or Zak phase, is directly proportional to the charge polarization in units of $\frac{e}{2\pi}$.

(e) *[Unseen but similar to derivations addressed in Lecture.]*
Rewriting the off-diagonal of $H_{\alpha,\beta}$ as $v + we^{ik} \equiv ze^{i\phi_k}$, we obtain

$$H_{\alpha,\beta} = \begin{pmatrix} 0 & ze^{-i\phi_k} \\ ze^{i\phi_k} & 0 \end{pmatrix}. \tag{3}$$

Hence, we directly verify that

$$|u_k, \pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm e^{i\phi_k} \\ 1 \end{pmatrix}.$$

For $v = w$ the energies are degenerate for $k = \pi$. For the gapped regions the Zak phase gives $\gamma = \frac{-1}{2}\oint \frac{d\phi_k}{dk}$. This is a winding number and cannot change and corresponds to the earlier results as $v < w$ implies $\delta t < 0$ and $v > w$ implies $\delta t > 0$. The Berry connection transforms as $A_k \to A_k + \partial_k \varphi_k$. This gives an extra piece to γ that reads $\int_0^{2\pi} \partial_k \varphi_k$ that is $\varphi(2\pi) - \varphi(0)$ which amounts to an arbitrary integer.

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2 Consider an electron with mass m and charge e moving in three dimensions under a uniform electric field $\mathbf{E}$ where the Hamiltonian is given by,

$$H = \frac{\mathbf{p}^2}{2m} - e\mathbf{E}\cdot\mathbf{r}\,.$$

Starting from $\mathbf{r}_i$ at $t_i = 0$, the probability amplitude for finding the electron at $\mathbf{r}_f$ at $t_f = t$ is given by the propagator

$$K(\mathbf{r}_f, t|\mathbf{r}_i, 0) = \theta(t)\langle \mathbf{r}_f|U(t,0)|\mathbf{r}_i\rangle\,,$$

for the time-evolution operator $U(t,0)$.

(a) Calculate how $K(\mathbf{r}_i, t|\mathbf{r}_f, 0)$ is related to $K(\mathbf{r}_f, t|\mathbf{r}_i, 0)$ and comment on your answer. [3]

(b) Consider the parity operator $\mathcal{P}|\mathbf{r}\rangle = |-\mathbf{r}\rangle$. Calculate whether the propagator satisfies reflection symmetry $K(\mathbf{r}_f, t|\mathbf{r}_i, 0) \overset{?}{=} K(-\mathbf{r}_f, t|-\mathbf{r}_i, 0)$. Compare this with the result for the free particle propagator $K_0(\mathbf{r}_f, t|\mathbf{r}_i, 0)$ for $H_0 = \mathbf{p}^2/2m$. [5]

(c) For the propagator in momentum space $\tilde{K}(\mathbf{p}_f, t|\mathbf{p}_i, 0)$, show that the probability of finding the electron with momentum $\mathbf{p}_f$ after time t vanishes unless $\mathbf{p}_f = \mathbf{p}_i + e\mathbf{E}t$. [6]

Now consider the electric field to be along the y-axis, $\mathbf{E} = E\hat{y}$.

(d) Write down the classical action and calculate the propagator for this electric field. [6]

$\Big[$*Hint: The free particle propagator is* $K_0(\mathbf{r}_f, t|\mathbf{r}_i, 0) = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} \exp\left\{i\frac{m}{2\hbar t}(\mathbf{r}_f - \mathbf{r}_i)^2\right\}$.$\Big]$

(e) Consider a wave packet of electrons with the same energy moving in the $(x-y)$-plane starting from point A at the origin as shown in the figure. The wave packet splits into two that travel along two different paths, ABC (path 1) and ADC (path 2), to finally meet at C with coordinates (d_x, d_y). Assuming that the size of the wave packet is much smaller than all other length scales, calculate the phase difference between the two paths and comment on it.

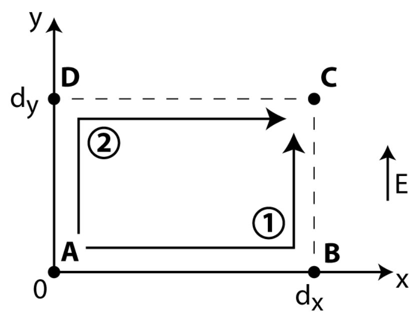

Figure 1:

[5]

Solution 2. (a) *[following calculations in lecture]*
Since the Hamiltonian is hermitian and real, the eigenstates can be found real, showing that the propagators are equal;

$$K(\mathbf{r}_f, t|\mathbf{r}_i, 0) = \sum_n \Psi_n(\mathbf{r}_f)\Psi_n^*(\mathbf{r}_i)e^{-i\epsilon_n t} = \sum_n \Psi_n^*(\mathbf{r}_f)\Psi_n(\mathbf{r}_i)e^{-i\epsilon_n t} = K(\mathbf{r}_i, t|\mathbf{r}_f).$$

(We set $\hbar = 1$.)

(b) *[Unseen but closely follows computations of the Handout]*
The reflected propagator can be written by using the parity operator as:

$$\begin{aligned} K(-\mathbf{r}_f, t|-\mathbf{r}_i, 0) &= \langle \mathbf{r}_f|\mathcal{P}e^{-iHt}\mathcal{P}|\mathbf{r}_i\rangle = \langle \mathbf{r}_f|e^{-iHt}|\mathbf{r}_i\rangle + \langle \mathbf{r}_f|[\mathcal{P}, e^{-iHt}]\mathcal{P}|\mathbf{r}_i\rangle \\ &= K(\mathbf{r}_f, t|\mathbf{r}_i, 0) + \langle \mathbf{r}_f|[\mathcal{P}, e^{-iHt}]\mathcal{P}|\mathbf{r}_i\rangle. \end{aligned}$$

Since $[\mathcal{P}, e^{-iHt}] \neq 0$ for $\mathbf{E} \neq 0$, the propagator does not have reflection symmetry as shown above. The free particle K_0 on the other hand is reflection symmetric.

(c) *[Unseen, but familiar from lecture]*
This naturally follows from the Heisenberg equation of motion for the momentum operator, and as expected classically the electric field is responsible for the momentum transfer.

$$[\hat{P}, e^{-iHt}] = e^{-iHt}e^{iHt}\hat{P}e^{-iHt} - e^{-iHt}\hat{P} = e^{-iHt}(\hat{P}_H(t) - \hat{P}),$$

where $\frac{\mathrm{d}\hat{P}_H}{\mathrm{d}t} = i[H, \hat{P}_H] = e\mathbf{E}$ so that $\hat{P}_H(t) = \hat{P} + e\mathbf{E}t$. Projecting on momentum basis for connecting to the propagator,

$$\begin{aligned} \langle \mathbf{p}_f|[\hat{P}, e^{-iHt}]|\mathbf{p}_i\rangle &= (\mathbf{p}_f - \mathbf{p}_i)\langle \mathbf{p}_f|e^{-iHt}|\mathbf{p}_i\rangle = \langle \mathbf{p}_f|e^{-iHt}(\hat{P}_H(t) - \hat{P})|\mathbf{p}_i\rangle \\ (\mathbf{p}_f - \mathbf{p}_i)\langle \mathbf{p}_f|, e^{-iHt}|\mathbf{p}_i\rangle &= e\mathbf{E}t\langle \mathbf{p}_f|e^{-iHt}|\mathbf{p}_i\rangle. \end{aligned}$$

So, for the momentum space propagator $\tilde{K}(\mathbf{p}_f, t|\mathbf{p}_i, 0) = \langle \mathbf{p}_f|e^{-iHt}|\mathbf{p}_i\rangle$ not to vanish, $\mathbf{p}_f = \mathbf{p}_i + e\mathbf{E}t$ must be.

(d) *[Familiar calculations in lecture, handout and examples]*
The equations of motions, $m\ddot{\mathbf{r}} = e\mathbf{E}\hat{y}$ and $\mathbf{r} = \mathbf{v}(0)t + \frac{e\mathbf{E}}{2m}t^2\hat{y}$. The classical action is,

$$S = \int_0^t \mathrm{d}t' \left(\frac{1}{2}mv^2 + e\mathbf{E}y(t')\right) = \frac{mr^2}{2t} + \frac{1}{2}etEy - \frac{e^2E^2t^3}{24m}.$$

The propagator can be written as

$$K(\mathbf{r}_f, t|\mathbf{r}_i, 0) = Ce^{\frac{i}{\hbar}S} = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{\frac{i}{\hbar}S} = K_0(\mathbf{r}_f, t|\mathbf{r}_i, 0)e^{i\frac{1}{2\hbar}etEy}e^{-i\frac{e^2E^2t^3}{24m\hbar}},$$

where the coefficient C is found by taking the free particle limit to reproduce K_0.

(e) *[Unseen]*
The classical action along the two paths are different, resulting in a phase difference in the probability amplitudes. By calculating the action along the two paths, we find the phase difference

$$\Delta\Phi = \frac{1}{\hbar}(S_{ABC} - S_{ADC}) = \frac{2e\mathbf{E}}{\hbar}d_y(t - t_1).$$

3 We consider a model for electron-phonon coupling given as $H = H_0 + H_1$, where

$$H_0 = \sum_{\mathbf{q},\mathbf{k}} \hbar\omega_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + \epsilon_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$$

$$H_1 = \sum_{\mathbf{q},\mathbf{k}} M_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{a}_{\mathbf{k}} (\hat{b}_{\mathbf{q}} + \hat{b}^\dagger_{-\mathbf{q}}).$$

Here the operators $\hat{b}^\dagger_{\mathbf{q}}$ describe the bosonic creation operator for phonons, that is lattice vibrations, with momentum $\mathbf{q}$ and frequency $\omega_{\mathbf{q}}$; and $\hat{a}^\dagger_{\mathbf{k}}$ denote creation operators for electrons with energy $\epsilon_{\mathbf{k}}$ and wave vector $\mathbf{k}$. We assume that H_1 is a weak coupling term that we can treat perturbatively in this problem.

(a) Give the fundamental algebraic relations that the phonon operators $\hat{b}^\dagger_{\mathbf{q}}$ and $\hat{b}_{\mathbf{q}}$, and electron operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ must obey due to their particle nature. [3]

(b) We may perform a similarity transformation $\tilde{H} = e^{-S} H e^{S}$. Assume S is of the same order as H_1 and remember that H_1 is treated perturbatively. Show that if S satisfies

$$H_1 + [H_0, S] = 0,$$

the first order term of $\tilde{H}$ when written as perturbative series vanishes, rendering only H_0 plus higher order terms. What is the second order term in the expansion in this case?

[*Hint: Recall that* $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + ...$]

[8]

(c) We assume that S has the form,

$$S = \sum_{\mathbf{q},\mathbf{k}} (\alpha \hat{b}^\dagger_{-\mathbf{q}} + \beta \hat{b}_{\mathbf{q}}) M_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{a}_{\mathbf{k}}.$$

Compute α and β such that the condition in part (*b*) is fulfilled. Write down the explicit form of S and give an interpretation for the denominator of α and β. [11]

[*Hint: Note that* $[AB, CD] = A[B,C]D + [A,C]BD + CA[B,D] + C[A,D]B$ *and that a similar (but not the same!) relation for* $[AB, CD]$ *in terms of anti-commutators can be derived.*]

(d) Show that the second order term in $\tilde{H}$, which is proportional to $|M_{\mathbf{q}}|^2$, has terms where the operator is of the form

$$\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{a}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}'-\mathbf{q}} \hat{a}_{\mathbf{k}'}.$$

What is the interpretation? [3]

Solution 3. (a) *[definition from notes]*
In the notes it is directly given that bosons have $[\hat{b}_{\mathbf{q}}, \hat{b}^\dagger_{\mathbf{q}'}] = \delta_{\mathbf{q},\mathbf{q}'}$, whereas all other commutators are zero. For fermions we have the same result in terms of anti-commutators.

(b) *[Unseen but closely follows computations of the Handout]*
We write $\tilde{H} = e^{-\alpha S}(H_0 + \alpha H_1)e^{\alpha S}$ to keep orders of α. Using $e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ...$ we thus obtain

$$\tilde{H} = H_0 + \alpha(H_1 + [H_0, S]) + \alpha^2([H_1, S] + \frac{\alpha^2}{2}[H_0, [H_0, S]].$$

This shows that when $H_1 + [H_0, S] = 0$ the first order vanishes and sets the second order to $\alpha^2/2[H_1, S]$.

(c) We use the form as given. Then we use two identities to calculate the commutator. Namely $[AB, C] = A[B, C] + [A, C]B$. Similarly, we rewrite $[AB, CD]$ as $[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B$.
As a result evaluating $[H_0, S] = [\hbar\omega_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{q}}\hat{b}_{\mathbf{q}}, S] + [\epsilon_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}, S]$ simply gives

$$\sum_{\mathbf{q},\mathbf{k}}(\alpha\hbar\omega_{-\mathbf{q}}\hat{b}^\dagger_{-\mathbf{q}} + \beta\hbar\omega_{\mathbf{q}}\hat{b}_{\mathbf{q}})M_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}_{\mathbf{k}} + \sum_{\mathbf{q},\mathbf{k}}(\alpha\hat{b}^\dagger_{-\mathbf{q}} + \beta\hat{b}_{\mathbf{q}})M_{\mathbf{q}}(\epsilon_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}_{\mathbf{k}} - \epsilon_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}_{\mathbf{k}})$$

As a result $H_1 + [H_0, S] = 0$ gives

$$H_1 + [H_0, S] = \sum_{\mathbf{q},\mathbf{k}}(\alpha'\hat{b}^\dagger_{-\mathbf{q}} + \beta'\hat{b}_{\mathbf{q}})M_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}_{\mathbf{k}},$$

where $\alpha' = 1 + \alpha(\hbar\omega_{-\mathbf{q}} + \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})$ and $\beta' = 1 + \beta(-\hbar\omega_{-\mathbf{q}} + \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})$. We demand $\alpha'\beta' = 0$, giving $\alpha = (-\hbar\omega_{-\mathbf{q}} - \epsilon_{\mathbf{k}+\mathbf{q}} + \epsilon_{\mathbf{k}})^{-1}$ and $\beta = (\hbar\omega_{-\mathbf{q}} - \epsilon_{\mathbf{k}+\mathbf{q}} + \epsilon_{\mathbf{k}})^{-1}$. We see these are the vertices of the interaction diagram and momentum conservation of energy and momentum as transfer.

(d) the term proportional to α^2 is $[H_1, S]$, see (b).We use the form as given. We rewrite $[AB, CD]$ as $[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B$.
When we evaluate $[\sum_{\mathbf{q},\mathbf{k}} M_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}_{\mathbf{k}}(\hat{b}_{\mathbf{q}} + \hat{b}^\dagger_{-\mathbf{q}}), \sum_{\mathbf{q}',\mathbf{k}'}(\alpha\hat{b}^\dagger_{-\mathbf{q}'} + \beta\hat{b}_{\mathbf{q}'})M_{\mathbf{q}'}\hat{a}^\dagger_{\mathbf{k}'+\mathbf{q}'}\hat{a}_{\mathbf{k}'}]$ the commutators of the phonons give delta functions that equate $\mathbf{q}$ and $\mathbf{q}'$. We see again conservation of momentum and energy. That is, the commuators involve products of a phonon part $(\alpha\hat{b}^\dagger_{-\mathbf{q}} + \beta\hat{b}_{\mathbf{q}})$ and electron part $\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}_{\mathbf{k}}$. We use the form as given and rewrite $[AB, CD]$ as $[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B$. The commutator of the commutators of the phonons give delta functions that equate $\mathbf{q}$ and $\mathbf{q}'$ and the electron oeprators involve commutators $\hat{a}^\dagger\hat{a}$ that are computed as in c. This reflects momentum conservation.

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4 Consider a system of three distinguishable spin-1/2 particles, which we denote with $+/-$ for up/down projections along the z-axis. Denote the Hilbert space as $\mathcal{H}$.

(a) Write down a basis for $\mathcal{H}$. [2]

(b) Give the expression for the entanglement S of a density operator ρ. Why is S zero for a pure state? [3]

(c) Write down an *entangled* state of these three spin-1/2 particles in which particles 2 and 3 are entangled, but the pair (2,3) is not entangled with particle 1. [5]

(d) Consider another entangled state,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|++-\rangle - |--+\rangle).$$

What is the reduced density operator ρ_2 obtained from tracing out the third particle staring from the pure state $|\Psi\rangle$? Is ρ_2 describing a pure or a mixed state? What is the probability that the first two particles are entangled? [7]

(e) A variation on the so-called W-state that occurs in quantum computing is the V-state given as,

$$|\Phi\rangle = \frac{1}{\sqrt{3}}(|++-\rangle + |+-+\rangle + |-++\rangle).$$

What is the reduced density operator $\tilde{\rho}_2$ for *any* two particles obtained from tracing out the third. Describe $\tilde{\rho}_2$. Is it a pure or a mixed state? What is the probability to find two remaining particles in an entangled state? [8]

Solution 4. (a) take the tensor product of up and down of the 3 states, giving a 8-dimensional Hilbert space parametrized by these vectors.

(b) $S = -\rho \log \rho$, given that eigenvalues are bounded by 1 and, when 1 only occurs once as pure state, a pure state with $\rho^2 = \rho$ implies $S = 0$.

(c) take e.g. the a Bell state $|\Psi\rangle \frac{1}{\sqrt{2}}(| + -\rangle - | - +\rangle$ and take the tensor product with $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle$, $|\Psi\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle$, then measuring the third has no consequence for the first two.

(d) Tracing out the third spin gives

$$\rho_2 = \frac{1}{2}| + +\rangle\langle + + | + \frac{1}{2}| - -\rangle\langle - - |$$

.
Whatever the projection of the third spin the reduced system is a product state with same probability either both up or down. These are product states, no entanglement. ρ_2 pertains to a mixture.

(e)For any spin the other two separate in a superposition as $| + -\rangle + | - +\rangle$ and a piece that has the same configuration. That is, focusing on the first spin, we have

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|+_1\rangle \otimes \sqrt{2}(\frac{1}{\sqrt{2}}| +_2 -_2\rangle + | -_2 +_2\rangle) + |-_1\rangle \otimes | +_2 +_3\rangle),$$

where the subscript denote the spins 1,2,3. Similarly for taking out the second spin, we have

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|+_2\rangle \otimes \sqrt{2}(\frac{1}{\sqrt{2}}| +_1 -_3\rangle + | -_1 +_3\rangle) + |-_2\rangle \otimes | +_1 +_3\rangle).$$

This evidently also provides for the 3rd spin,

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|+_3\rangle \otimes \sqrt{2}(\frac{1}{\sqrt{2}}| +_1 -_2\rangle + | -_1 +_2\rangle) + |-_3\rangle \otimes | +_1 +_2\rangle).$$

If we denote $\frac{1}{\sqrt{2}}| + -\rangle + | - +\rangle)$ as Φ we see that the reduced density matrix is of the form

$$\tilde{\rho}_2 = \frac{1}{3}| + +\rangle\langle + + | + \frac{2}{3}|\Phi\rangle\langle\Phi|$$

. The inetrpretation is the same as before, although now we can collapse with $\frac{2}{3}$ probablity to a reduced system that is the Bell state Φ, which is entangled.
