UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Wednesday 24 April 2024, 10.30 to 12.30

THEORETICAL PHYSICS 2

Answer ***all four*** *questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains 4 sides, excluding this one, and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof. (The booklet is available for separate download.)

V7.1

A

1 (a) Consider a Hermitian eigensystem $H(\mathbf{g})|u_n(\mathbf{g})\rangle = E_n(\mathbf{g})|u_n(\mathbf{g})\rangle$, which depends parametrically on g^μ, $\mu = 1, ..., N$. Starting from the overlap of two infinitessimally close states in parameter space,

$$\langle u_n(\mathbf{g})|u_n(\mathbf{g}+\delta\mathbf{g})\rangle = |\langle u_n(\mathbf{g})|u_n(\mathbf{g}+\delta\mathbf{g})\rangle| \cdot e^{i\boldsymbol{\mathcal{A}}_n(\mathbf{g})\cdot\delta\mathbf{g}},$$

show that, upon leading order in $\delta\mathbf{g}$, $\mathcal{A}_n^a = \frac{\langle u_n(\mathbf{g})|\nabla_g^a u_n(\mathbf{g})\rangle - \langle \nabla_g^a u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}{2i\langle u_n(\mathbf{g})|u_n(\mathbf{g})\rangle}$. Assuming that the states are normalised, show this gives the Berry potential,

$$\mathcal{A}_n^a = -i\langle u_n(\mathbf{g})|\nabla_g^a u_n(\mathbf{g})\rangle. \tag{1}$$

[6]

(b) We now take a specific class of Hamiltonians of the above form,

$$H(\mathbf{q}) = v(q_x\sigma_x + q_y\sigma_y + q_z\sigma_z),$$

where $\mathbf{q}$ is assumed to be an effective momentum, σ_i are the Pauli matrices and v is a velocity. What happens at $\mathbf{q} = 0$? Show that $H(\mathbf{q})$ has two eigenenergies $E_\pm = \pm v|\mathbf{q}|$. [3]

(c) Show that the eigensstate $|+\rangle$, with $H|+\rangle = E_+|+\rangle$, is given by

$$|+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{pmatrix},$$

where we rewrote $(q_x, q_y, q_z) = (|\mathbf{q}|\sin\theta\cos\varphi, |\mathbf{q}|\sin\theta\sin\varphi, |\mathbf{q}|\cos\theta)$. Give also an expression for $|-\rangle$, where $H|-\rangle = E_-|-\rangle$. [6]

(d) Show that the Berry potential $\mathcal{A}_+^a = -i\langle +|\nabla^a +\rangle = \frac{\sin^2(\theta/2)\hat{\boldsymbol{\varphi}}}{|\mathbf{q}|\sin\theta}$ and that the Berry curvature, $\boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}}_+$, equates to $\frac{\hat{\boldsymbol{q}}}{2|\mathbf{q}|^3}$. [5]

(e) What is the result when we integrate the Berry curvature over a sphere of constant $\mathbf{q}$? Give an interpretation. How does the result change when we would instead consider the Hamiltonian $\tilde{H}(\mathbf{q}) = v(q_x\sigma_x + q_y\sigma_y - q_z\sigma_z)$? Motivate your answer [a calculation is not directly needed]. How do the results change when we make a gauge transformation, $|+\rangle \to e^{i\beta(\mathbf{q})}|+\rangle$? [5]

2 Consider a very thin and long wire of length L and width W, smoothly connected to a reservoir of electrons on both ends. The reservoirs at the left and right end are kept at different electrostatic potentials $V = V_R - V_L$.

(a) Write down the energy eigenvalues and eigenstates of the electron wave-functions in the wire.

[*hint: You can assume a rectangular shaped wire and combine the quantum particle in an infinite well problem for the transverse direction y with a free-particle in the longitudinal direction x.*] [4]

(b) For each eigenstate calculate the electrical current density $j_{n,k}(x,y)$ and current $\mathcal{I}_{n,k}$. Here n, k are the eigenstate labels. The total current through the wire is

$$I = 2\frac{L}{2\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dk \mathcal{I}_{n,k}[f(\epsilon_{n,k} + eV_L)\theta(k) + f(\epsilon_{n,k} + eV_R)\theta(-k)], \quad (2)$$

where $f(x)$ is the Fermi distribution function and $\theta(x)$ is the Heaviside step function. On physical grounds, justify the presence of these two functions and the first factor of 2 on the RHS of Eq. (2) . [6]

(c) Now take the zero temperature limit, and show that the conductance of the wire is given by $G = 2Ne^2/h$, where N is the number of occupied energy levels n (also referred to as open channels). Why is conductivity not infinite? Where is the associated energy loss happening?

[*hint: For the first part, it is convenient to perform integral over energy instead of k. For this try to express $\mathcal{I}_{n,k}$ as a derivative of energy*] [6]

(d) Now consider an impurity in the middle of the wire through which an electron in state n is reflected by probability r and transmitted by a probability t. Moreover, assume that the reflection and transmission is diagonal in n, *i.e.* the quantum number n remains unchanged before and after scattering. Obtain the modified expression for total current and show that the conductance is modified to $G = 2N|t|^2e^2/h$. [6]

(e) Now assume that the transmission and reflection probabilities are $t_{n,n'}$ and $r_{n,n'}$ for scattering between two channels with quantum number n and n', write down the generalized expression for conductance. You will get full marks even if you just guess the correct answer from what you obtained in (d) without any derivation. [3]

3 Consider spinless particles in a magnetic field described by the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}(\mathbf{r}))^2,$$

where we set the speed of light $c = 1$, $\mathbf{p}$ represents momentum and $\mathbf{A}(\mathbf{x})$ is the gauge potential.

(a) Give an expression for the velocity $\mathbf{v}$.

[*hint: Recall* $[AB, C] = A[B, C] + [A, C]B$ *and* $[A, BC] = B[A, C] + [A, B]C$.] [5]

(b) We now take a specific gauge $\mathbf{A}(\mathbf{r}) = \frac{1}{2}(-By, Bx, 0)$, where B is the strength of the magnetic field. Setting $e = \hbar = B = m = 1$ we then get following Hamiltonian

$$H_{symm} = \frac{1}{2}[(-i\frac{\partial}{\partial x} - \frac{y}{2})^2 + (-i\frac{\partial}{\partial x} + \frac{x}{2})^2].$$

Show that H_{symm} can be written in quantised form as

$$H_{symm} = \hat{a}^\dagger \hat{a} + \frac{1}{2},$$

where $\hat{a} = \frac{1}{\sqrt{2}}[(\frac{x}{2} + \frac{\partial}{\partial x}) - i(\frac{y}{2} + \frac{\partial}{\partial y})]$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}[(\frac{x}{2} - \frac{\partial}{\partial x}) + i(\frac{y}{2} - \frac{\partial}{\partial y})]$. Also verify that $[\hat{a}, \hat{a}^\dagger] = 1$. What is the interpretation of acting with $\hat{a}^\dagger$ on the vacuum state? [6]

We now consider turning the magnetic field off and switching on interactions. In second quantised form the system is then described by a Hamiltonian that reads

$$H_{int} = \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})[\frac{-\hbar^2}{2m}\nabla^2]\hat{\psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')V(\mathbf{r}, \mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}'),$$

where $\hat{\psi}^\dagger(\mathbf{r})$ are the field operators and $V(\mathbf{r}, \mathbf{r}')$ is a two-body interaction term.

(c) Write down the commutation/anti-commutation relations for the field operators $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ when the system is generally composed of Fermions or Bosons. [3]

(d) Define the total angular momentum J. Assuming that we have Fermions and a two-body interaction term of the form $V(\mathbf{r}, \mathbf{r}') = V(|\mathbf{r} - \mathbf{r}'|)$. Show that total angular momentum J is conserved. Here you may use that $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ vanish for large $\mathbf{r}$. You may also use that V derivatives thereof are symmetric in $\mathbf{r}$.

[*hint 1: Recall* $[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B$ *and* $\frac{d|\mathbf{r}|}{d\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$.]

[*hint 2: Also note that the boundary conditions allow for integration by parts.*] [8]

(e) What does the result imply for the "Landau levels" obtained acting with $\hat{a}^\dagger$ on the vacuum when such interaction terms are present? [3]

4 Consider the Hong-Ou-Mandel setup. Precisely, consider a 50:50 beam splitter (i.e. each incident photon has an equal probability of getting reflected or transmitted from the beam splitter), with two input modes and two output modes. Two identical photons are simultaneously incident in the two input modes (one in each mode).

(a) Draw the figure for all possible experimental outcomes. Taking into account the unitarity of the scattering matrix and assuming real-valued reflection and transmission amplitudes, assign appropriate overall signs for all possible outcomes. [4]

(b) Write down the two-photon states in the input and in the output modes. [*hint: The most convenient is to write Fock states in the mode basis*]. For the input and output states, write down the density matrices. Take the partial trace over one of the mode and obtain the reduced density matrices. Then explicitly calculate the entanglement entropies to show that the experiment generates entanglement in the output modes. [5]

(c) Now consider a three-qubit state known as the GHZ state

$$\psi_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Write down the one and two particle reduced density matrices by taking partial traces (You can choose to trace out particle 2 and 3 to calculate one particle reduced density matrix and trace out particle 1 to calculate two particle reduced density matrix). Calculate the entanglement entropies. Is the mutual entanglement equal? [5]

(d) By using the two particle reduced density matrix, show that indeed the GHZ state is highly entangled. [*hint: You can show that by considering a small perturbation to the GHZ state by mixing it with a three-qubit state of your choice and showing that entanglement entropy decreases as mixing is increased.*] [6]

(e) Now consider another highly entangled three qubit state, called the W state

$$\psi_W = \frac{1}{\sqrt{3}}(|001\rangle + |110\rangle + |100\rangle).$$

Trace out one of the particle and calculate the entanglement entropy. Compare it with the result of the GHZ state. Which state is more entangled? [5]