NATURAL SCIENCES TRIPOS Part II

Wednesday 16 January 2019 10:30am to 12:30pm

THEORETICAL PHYSICS I

Answer **all four** *questions.*

The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate.

The paper contains four sides and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.

1 The action for a system consisting of a polarizable medium moving relativistically in an electromagnetic field is given by

$$S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^{\mu\nu} F_{\mu\nu} - \frac{1}{2\kappa} u_\mu u_\nu M^{\lambda\mu} M^{\nu}{}_{\lambda} + \frac{1}{4\chi} (g_{\mu\rho} + u_\mu u_\rho)(g_{\nu\sigma} + u_\nu u_\sigma) M^{\mu\nu} M^{\sigma\rho} \right],$$

where $F^{\mu\nu}$ is the usual electromagnetic field strength tensor, $g^{\mu\nu}$ is the usual Minkowski metric, $M^{\mu\nu}$ is an antisymmetric tensor describing the polarization of the medium, u^μ is the 4-velocity of the medium, and κ and χ are physical constants.

(a) Show that the equation of motion for the electromagnetic gauge potential, A_μ, is given by

$$\partial_\nu F^{\mu\nu} = \partial_\nu M^{\mu\nu}.$$

[5]

[partly book work and partly unseen] The usual variation of the Maxwell Lagrangian $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ yields a term $\partial_\nu F^{\mu\nu}$ in the equation of motion. For the other term involving A_μ, it is easiest to integrate by parts to move the derivative onto the polarization tensor. Using antisymmetry of $M^{\mu\nu}$, one reduces the term to $M^{\mu\nu} \partial_\nu A_\mu$, whence the equation of motion is just $\partial_\nu F^{\mu\nu} = \partial_\nu M^{\mu\nu}$.

(b) Using the method of Lagrange multipliers to account for the antisymmetric nature of $M^{\mu\nu}$, show that the equation of motion for the antisymmetric polarization tensor, $M^{\mu\nu}$ is given by

$$0 = F_{\mu\nu} - \frac{1}{\kappa}(u_\nu u_\rho M^{\rho}{}_{\mu} - u_\mu u_\rho M^{\rho}{}_{\nu}) - \frac{1}{\chi}\left[M^{\rho\sigma}(g_{\rho\mu} + u_\rho u_\mu)(g_{\sigma\nu} + u_\sigma u_\nu) \right].$$

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[10]

[unseen calculation] We use the method of Lagrange multipliers to account for the fact that $M^{\mu\nu}$ is antisymmetric. Thus, we add to the Lagrangian a term $\lambda_{\mu\nu}(M^{\mu\nu} + M^{\nu\mu})$, where $\lambda_{\mu\nu}$ are Lagrange multipliers. This term adds a piece $\lambda_{\mu\nu} + \lambda_{\nu\mu}$ to the equation of motion for $M^{\mu\nu}$, so we can easily remove the lagrange multipliers from the equations of motion by antisymmetrizing the equations of motion with respect to μ and ν. In this way, a straightforward but tedious derivation yields

$$0 = F_{\mu\nu} - \frac{1}{\kappa}(u_\nu u_\rho M^\rho{}_\mu - \{\mu \leftrightarrow \nu\}) - \frac{1}{\chi}\left[M^{\rho\sigma}(g_{\rho\mu} + u_\rho u_\mu)(g_{\sigma\nu} + u_\sigma u_\nu)\right]$$

Note: An answer which does not use Lagrange multipliers, or does not otherwise explicitly take into account the fact that $M^{\mu\nu}$ is antisymmetric, will be penalized.

(c) Defining $P_i \equiv M^{i0}$ and $M_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$, respectively, express the interaction term $\frac{1}{2}M^{\mu\nu}F_{\mu\nu}$ in terms of the usual electric and magnetic field vectors and thus give an interpretation of P_i and M_i. [4]

[unseen calculation] Using the usual relations, we get $\frac{1}{2}M^{\mu\nu}F_{\mu\nu} = P \cdot E + M \cdot B$, whence we see that P represents the electric polarization (per unit volume) and M the magnetic polarization (per unit volume). Differing signs of either term correspond simply to different conventions so will not be penalised.

(d) Show that, in the rest frame, P_i and M_i are proportional to the electric and magnetic fields, respectively, and find the constants of proportion. [4]

[partly book work and partly unseen] Using $u^\mu = (1, 0, 0, 0)$ in the rest frame, we get $P = \kappa E, M = \chi B$.

(e) What properties of the medium do the constants κ and χ describe? [2]

[bookwork] By inspection, κ and χ represent (roughly) the electric and magnetic susceptibilities of the medium, respectively.

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2 Describe what is meant by a *holonomic constraint* in Lagrangian classical mechanics and give an example of a system with a holonomic constraint. [4]

[bookwork] A holonomic constraint is one that can be expressed in terms of generalized co-ordinates and time alone. An example is the system below, if oscillations are small enough.

A uniform cylinder of mass m and radius r rolls without slipping inside a larger cylinder of radius R. The angles θ and ϕ are defined as in the Figure, where the black dots coincide when the rolling cylinder is at its lowest point. By resolving the motion into the motion of the centre of mass and motion about the centre of mass, show that the kinetic energy may be written as

$$\frac{m}{2}(R-r)^2\dot{\theta}^2 + \frac{m}{4}r^2(\dot{\phi}-\dot{\theta})^2$$

[5]

[unseen] The motion of the centre of mass is purely rotational, in a circle of radius $R-r$ with angular velocity $\dot{\theta}$. The rotation of the cylinder about its centre of mass *in the rest frame* is with angular velocity $\dot{\phi}-\dot{\theta}$. The moment of inertia is $mr^2/2$. The total kinetic energy is therefore

$$\frac{m}{2}(R-r)^2\dot{\theta}^2 + \frac{m}{4}r^2(\dot{\phi}-\dot{\theta})^2$$

.

Show that the no slip condition yields a holonomic constraint. [3]

[unseen] The no slip condition forces $r\phi = R\theta$ which is a holonomic constraint.

By writing down the Lagrangian, show that the system can exhibit small oscillations and find the corresponding oscillation frequency. [6]

[unseen] To the kinetic energy, we must add the potential energy $-mg(R-r)\cos\theta$. We also have the no slip condition, which forces $r\phi = R\theta$. We may either include this as a lagrange multiplier or solve directly. With the latter approach, we get

$$L = \frac{3}{4}m(R-r)^2\dot{\theta}^2 + mg(R-r)\cos\theta$$

from which we read off the angular frequency $\omega^2 = \frac{2g}{3(R-r)}$.

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Discuss whether or not there is a frictional force present and whether or not the energy of the system is conserved. [3]

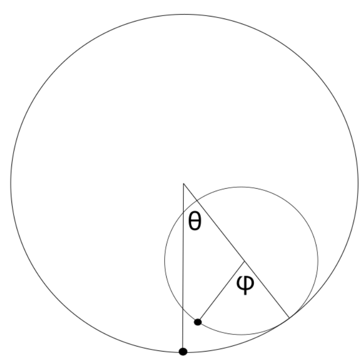
[bookwork] There must be a frictional force present, because the cylinder rolls without slipping. However, this force does no work because the motion is always normal to it. Hence the energy is conserved, and indeed the Lagrangian, being explicitly independent of time, has conserved energy by Noether's theorem.

Discuss whether or not angular momentum is conserved. [2]

[bookwork] No, and indeed the system is not invariant under rotations (e.g. because of the presence of gravity).

Discuss whether or not the system is holonomic for all possible motions. [2]

[unseen] For large motions, the small cylinder can lose contact with the larger one. This cannot be expressed in terms of holonomic constraints.



3 A Bose-Einstein condensate can described by the Lagrangian density

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) - \frac{\hbar^2}{2M} \boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi + \mu|\psi|^2 - \frac{1}{2} g|\psi|^4$$

for the complex field $\psi(\boldsymbol{r}, t)$, with $g, \mu > 0$.

(a) Show that this theory has symmetry under global changes of the phase of $\psi(\boldsymbol{r}, t)$, but does not have symmetry under local phase changes. [3]

[bookwork] *Since all terms of products of equal numbers of ψ^* and ψ, the action is clearly invariant under the global phase change $\psi \to e^{i\epsilon}\psi$. However, if ϵ depends on space or time, then the space- or time-derivatives would generate extra terms not present initially.*

(b) Show that the Hamiltonian density is

$$\mathcal{H} = \frac{\hbar^2}{2M} \boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - \mu|\psi|^2 + \frac{1}{2} g|\psi|^4 \, .$$

Hence show that the groundstate, ψ_0, breaks the global symmetry. [5]

[bookwork] *Standard bookwork, starting from*

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*}\dot{\psi}^* + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} - \mathcal{L}$$

leads to the result.

To minimize $\mathcal{H}$, the groundstate must be spatially uniform.
The minimum of $-\mu|\psi|^2 + \frac{1}{2} g|\psi|^4$ is at

$$|\psi|^2 = \frac{\mu}{g} \Rightarrow \psi = \sqrt{\frac{\mu}{g}} e^{i\theta}$$

Hence it breaks the symmetry: any choice of θ gives a valid groundstate.

(c) By writing $\psi(\boldsymbol{r}, t) = \psi_0 + \chi(\boldsymbol{r}, t)$, with ψ_0 real, show that, keeping all terms up to second order in χ, the Lagrangian density may be written [7]

$$\mathcal{L} \simeq \frac{i\hbar}{2}\left[\chi^* \frac{\partial \chi}{\partial t} - \chi \frac{\partial \chi^*}{\partial t}\right] - \frac{\hbar^2}{2M} \boldsymbol{\nabla}\chi^* \cdot \boldsymbol{\nabla}\chi - \frac{1}{2}\mu(\chi + \chi^*)^2$$

[familiar ideas, but different temporal dependence] *Insert $\psi(\boldsymbol{r}, t) = \psi_0 + \chi(\boldsymbol{r}, t)$ and expand up to second order in χ.*

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Use must be made of: $|\psi|^2 = \mu/g$ *to simplify the terms.*

There is an overall constant and a term $\frac{1}{2}\mathrm{i}\hbar(\psi_0^*\dot{\chi} - \psi_0\dot{\chi}^*)$ *which must be dropped to get the stated result. Candidates are expected to point out that terms that are total derivatives can be ignored.*

(d) Derive the Euler-Lagrange equations for χ and χ^*. By writing $\chi = \chi_1 + \mathrm{i}\chi_2$, or otherwise, show that there exist excitations with angular frequency ω and wavevector $\boldsymbol{k}$ connected via [8]

$$\omega = \sqrt{\frac{|\boldsymbol{k}|^2}{2M}\left(\frac{\hbar^2|\boldsymbol{k}|^2}{2M} + 2\mu\right)}.$$

[unseen] *The E-L equations can be found by using* χ *and* χ^* *as independent variables:*

$$\begin{aligned} \mathrm{i}\hbar\dot{\chi} &= -\frac{\hbar^2}{2M}\nabla^2\chi + \mu(\chi + \chi^*) \\ -\mathrm{i}\hbar\dot{\chi}^* &= -\frac{\hbar^2}{2M}\nabla^2\chi^* + \mu(\chi + \chi^*) \end{aligned}$$

Take the sum and difference to get the real and imaginary parts χ_1 *and* χ_2

$$\begin{aligned} -\hbar\dot{\chi}_2 &= -\frac{\hbar^2}{2M}\nabla^2\chi_1 + 2\mu\chi_1 \\ \hbar\dot{\chi}_1 &= -\frac{\hbar^2}{2M}\nabla^2\chi_2 \end{aligned}$$

For a plane wave $e^{-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$*, we have*

$$\begin{aligned} +\mathrm{i}\hbar\omega\chi_2 &= \frac{\hbar^2|\boldsymbol{k}|^2}{2M}\chi_1 + 2\mu\chi_1 \\ -\mathrm{i}\hbar\omega\chi_1 &= \frac{\hbar^2|\boldsymbol{k}|^2}{2M}\chi_2 \end{aligned}$$

Hence,

$$\begin{aligned} -\mathrm{i}\hbar\omega\chi_1 &= \frac{\hbar^2|\boldsymbol{k}|^2}{2M}\left[\frac{\hbar^2|\boldsymbol{k}|^2}{2M} + 2\mu\right]\frac{1}{\mathrm{i}\hbar\omega}\chi_1 \\ (\hbar\omega)^2 &= \frac{\hbar^2|\boldsymbol{k}|^2}{2M}\left[\frac{\hbar^2|\boldsymbol{k}|^2}{2M} + 2\mu\right] \end{aligned}$$

giving the requested result.

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(e) Comment on the form of this mode dispersion in connection with Goldstone's theorem. [2]

[standard concept] *There is a broken global continuous symmetry, so one expects there to be a gapless mode according to Goldstone's theorem. The mode derived indeed has a vanishing gap at* $k \to 0$.

4 The Landau free energy describing a certain magnetic material is

$$f(m) = a(T - T_c)m^2 + \frac{1}{2}b\,m^4 + \frac{1}{3}c\,m^6$$

where the order parameter m is real, and $c > 0$.

(a) Explain the physical meaning of the Landau free energy and the order parameter. What symmetries does the above theory exhibit? [5]

[bookwork] *The order parameter characterises the phase transition: it vanishes in the disordered phase and is non-zero in the ordered phase; it can distinguish between different broken symmetry phases.*

The Landau free energy represents a coarse-grained description of the system, valid close to a critical point where the order parameter is small.

(b) For $b > 0$, show that there is a continuous phase transition to an ordered phase for $T < T_c$. Determine the value of the critical exponent β, defined by $m \propto (T_c - T)^\beta$ for T close to T_c in the ordered phase. [8]

[bookwork] *Close to the transition m is small so we can neglect cm^6. This reduces to the model (studied in handout and lectures) $f(m) \simeq a(T - T_c)m^2 + \frac{1}{2}b\,m^4$.*

The free energy has stationary points at $m = 0$ and $m^2 = a(T_c - T)/b$.

For $T > T_c$ the only (real) solution is $m = 0$, the disordered phase.

For $T < T_c$ the free energy minima are at $m = \pm\sqrt{a(T_c - T)/b}$. (That these have lower free energy than $m = 0$ should be stated explicitly.)

(c) Consider now $b < 0$. By sketching the free energy for $T = T_c$, or otherwise, show that the phase transition is now discontinuous. [4]

[extension of known ideas to new setting] *At $T \leq T_c$, the $m = 0$ solution becomes unstable. However, at $T = T_c$, there is already another (global) minimum at nonzero m. (This could be shown on a sketch.) Hence, the global energy minimum must have moved away from $m = 0$ at some higher $T > T_c$. At that point there was a jump in m from zero to a non-zero value: i.e. a discontinuous phase transition.*

(d) Determine the critical temperature T_c^* at which this discontinuous transition occurs. [*Hint: it can be helpful to work in terms of $x = m^2$.*] [6]

[unseen] *At the transition point, T_c^*, the free energy of $m = 0$ equals that at the new, nonzero, value m^* to which the system jumps. The free energy at $m = 0$ is $f(0) = 0$, so the new minimum also has $f(m^*) = 0$.*

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To analyse the transition, follow the hint and work in terms of $x = m^2$

$$f(x) = \alpha x + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 = x(\alpha + \frac{1}{2}bx + \frac{1}{3}cx^2)$$

The minimum corresponding to $m = 0$ corresponds to the root at $x = 0$. The second minimum to which it jumps at the transition can be found by noting that this corresponds to the situation where the two roots of $(\alpha + \frac{1}{2}bx + \frac{1}{3}x^2)$ coincide (i.e. the minimum of $f(x)$ just touches $f = 0$). From the formula for the roots of a quadratic equation, that requires:

$$(b/2)^2 - 4\alpha(c/3) = 0 \quad \Rightarrow \quad \alpha = \frac{3}{16}\frac{b^2}{c}$$

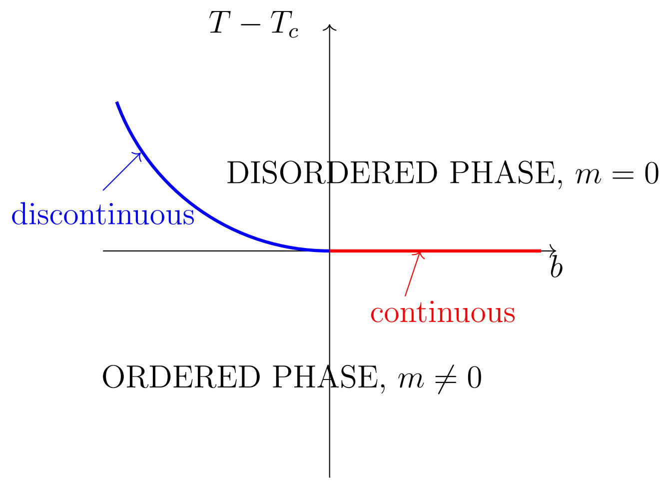
Hence

$$T_c^* = T_c + \frac{3}{16}\frac{b^2}{ac}$$

[Credit will be given for other approaches that would lead to this result, e.g. by explicitly finding the minima of $f(m)$ via differentiation.]

(e) Hence, or otherwise, sketch the phase diagram as a function of $T - T_c$ and of b, with both of these parameters ranging from negative to positive values. Be clear to distinguish continuous and discontinuous transitions. [2]

[unseen]



The answer could also be obtained from physical reasoning that T_c^*, at $b < 0$, should increase as $|b|$ becomes larger (since nonzero $|b|$ makes the minimum to which the system will jump have lower free energy).

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