UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 19 January 2024 2 pm to 4 pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains* **5** *sides, including this one and is accompanied by a booklet giving values of constants and containing mathematical formulæ which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 Two planets, of masses m_1 and m_2 and negligible size, interact via gravity.

(a) By writing down the lagrangian in the centre-of-mass frame and solving the Euler-Lagrange equations of motion, show that the planets may undergo circular motion at any radius of separation r with constant angular frequency ω given by $\omega^2 = G(m_1 + m_2)/r^3$. [5]

A satellite, of mass m_3 and negligible size, is added to the system. You may assume that it has a negligible effect on the motion of the planets and that it moves in the plane of their circular motion.

(b) Show that by choosing suitable coordinates x and y in the frame of reference in which the planets are stationary, the lagrangian for the satellite may be written as

$$L = \frac{1}{2} m_3 \left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 \right] + G m_3 \left[\frac{m_1}{r_1} + \frac{m_2}{r_2} \right],$$

where you should determine r_1 and r_2 in terms of x, y, m_1, m_2, and r. [4]

(c) Find the equation of motion for x and show that the equation of motion for y is given by

$$m_3(\ddot{y} + 2\omega\dot{x} - \omega^2 y) = -G m_3 \left[\frac{m_1 y}{r_1^3} + \frac{m_2 y}{r_2^3} \right].$$

[4]

(d) Find the locations of the two points away from the line joining the two planets at which the satellite may be stationary with respect to the planets. [4]

(e) By means of a graphical method, find out how many points there are on the line joining the two planets at which the satellite may be stationary with respect to the planets. [6]

(f) Draw a sketch showing the planets and all the possible stationary points. [2]

2 A relativistic real scalar field ϕ in 1+1-dimensional spacetime, in units where $c = 1$ with co-ordinates $x^\mu = (t, x)$ and Minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, has lagrangian density given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + m^2\cos\phi,$$

where $m^2 > 0$.

(a) Write down the Euler-Lagrange equation of motion for ϕ. [2]

(b) Write down the conserved energy-momentum tensor, in terms of ϕ and its derivatives, and discuss whether it is symmetric. [4]

(c) Find the values of α such that the transformation $\phi(x^\mu) \mapsto \phi(x^\mu) + \alpha$ is a symmetry; if there is a corresponding conserved current, find it. [2]

(d) Show that there are no plane wave solutions to the Euler-Lagrange equation. [2]

(e) Show that a solution depending on x^μ only via the combination $\tau = t - x/v$, for some v such that $-1 < v < 1$, must satisfy the equation

$$\frac{d^2\phi}{d\tau^2} = A\sin\phi,$$

where A is a constant that you should determine in terms of m and v. [2]

(f) Find the values of B and C for which there is a solution of the form

$$\phi = 4\arctan\exp\left(B\tau + C\right)$$

and draw a sketch of such a solution, explaining why it is physically reasonable. [8]

(g) Identify as many symmetries of the system as you can and discuss whether these can be used to find new solutions from the solutions you have already found. [5]

(TURN OVER

3 Consider a particle moving in $2+1$ spacetime dimensions described by a hamiltonian H. The differential equation describing the propagation of this particle from position $\boldsymbol{r}$ at time t to position $\boldsymbol{r'}$ at time t' is given in terms of the Green function $G(\boldsymbol{r}, \boldsymbol{r'}, t, t')$ by

$$\left(i\hbar\frac{\partial}{\partial t} - H\right) G(\boldsymbol{r}, \boldsymbol{r'}, t, t') = \delta^2(\boldsymbol{r} - \boldsymbol{r'})\delta(t - t').$$

The Green function is to be studied using the Fourier transforms

$$G(\boldsymbol{r}, \boldsymbol{r'}, E) = \int dt \; \exp(iE(t - t')/\hbar)G(\boldsymbol{r}, \boldsymbol{r'}, t, t'),$$
$$G(\boldsymbol{k}, E) = \iint d^2\boldsymbol{r} \; \exp(i\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r'}))G(\boldsymbol{r}, \boldsymbol{r'}, E).$$

(a) Describe what it means for a Green function to be causal and how causality may be implemented using Fourier transforms. [5]

(b) Suppose that the particle is of mass m and propagates non-relativistically and freely, such that the Hamiltonian is $H = -\frac{\hbar^2\nabla^2}{2m}$. Show that, with causality, the Green function $G(\boldsymbol{r}, \boldsymbol{r'}, E)$ can be written, for $\boldsymbol{r'} = \boldsymbol{r}$ and $E > 0$, as

$$G(\boldsymbol{r}, \boldsymbol{r'} = \boldsymbol{r}, E > 0) = \lim_{\delta \to 0} G_\delta,$$

where

$$G_\delta = \alpha \int_0^\infty dk \left(\frac{1}{k - k_+ - i\delta} + \frac{1}{k + k_+ + i\delta}\right), \qquad (1)$$

δ is real, positive, and arbitrarily small, and α and k_+ are constants that you should determine. [7]

(c) Use the formula

$$\rho(E) = \frac{-1}{2\pi i} \lim_{\delta \to 0} \left[G_{|\delta|} - G_{-|\delta|}\right],$$

where G_δ is defined for both positive and negative values of δ by equation (1) above, to compute the density of states $\rho(E)$ for $E > 0$. [7]

(d) In graphene, the hamiltonian can be written in terms of $\boldsymbol{k}$ as $H = \hbar v|\boldsymbol{k}|$, where v is a real positive constant. Calculate the density of states $\rho(E)$ for $E > 0$. [6]

[*Hint: For real x and real positive y, the imaginary part of* $\lim_{y \to 0} \frac{1}{x+iy}$ *is given by* $-\pi\delta(x)$.]

4 This question involves using a microscopic model to study phase transitions in a system in thermal equilibrium at temperature T.

(a) Describe the meaning of the term *order parameter* in the context of phase transitions and give an example of such an order parameter. [4]

The Dicke model describes the interaction of light, modelled by a real-valued field α, with a number N of two-state atomic emitters. The partition function of the model is given by

$$Z(\alpha) = e^{-\beta\omega\alpha^2} \left(\mathrm{Tr}\, e^{-\beta h(\alpha)}\right)^N .$$

Here, $h(\alpha)$ is the 2×2-matrix given by

$$h(\alpha) = \frac{\Delta}{2}\sigma_z + \frac{2g}{\sqrt{N}}\alpha\sigma_x,$$

while $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the usual Pauli matrices, $\beta = 1/kT$, and k, ω, Δ, and g are real positive constants.

(b) Show that the free energy, $F(\alpha) = -\frac{1}{\beta}\ln\left(Z(\alpha)\right)$, can be written as

$$F(\alpha) = b\alpha^2 - c\ln\Big(2\cosh(\beta E(\alpha))\Big),$$

where you should determine b, c and $E(\alpha)$ in terms of the parameters already given. [7]

(c) Explain why the free energy always has either one minimum at $\alpha = 0$ or two minima, each at non-zero values of α (you need not find the explicit values of α). [7]

(d) Show that, when $g^2/\omega\Delta$ exceeds a certain value (which you should determine), there is a phase transition and find the temperature at which the transition occurs. [7]

END OF PAPER