UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 19 January 2024 2 pm to 4 pm

THEORETICAL PHYSICS I

Attempt **all 4** *questions. The approximate number of marks allotted to each part of a question is indicated in the right margin. The paper contains* **13** *sides, including this one and is accompanied by a booklet giving values of constants and containing mathematical formulæ which you may quote without proof.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 Two planets, of masses m_1 and m_2 and negligible size, interact via gravity.

(a) By writing down the lagrangian in the centre-of-mass frame and solving the Euler-Lagrange equations of motion, show that the planets may undergo circular motion at any radius of separation r with constant angular frequency ω given by $\omega^2 = G(m_1 + m_2)/r^3$. [5]

[Seen] In the centre of mass frame, we have that

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + GM\mu/|\boldsymbol{r}|,$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ and $M = m_1 + m_2$. In polar co-ordinates (r, θ), where $r = |\boldsymbol{r}|$, the Euler-Lagrange equation of motion for r is

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - GM\mu/r^2,$$

so for circular motion with $\dot{r} = 0, \dot{\theta} = \omega$ this reduces to

$$0 = r\omega^2 - GM\mu/r^2$$

giving the desired result.

A satellite, of mass m_3 and negligible size, is added to the system. You may assume that it has a negligible effect on the motion of the planets and that it moves in the plane of their circular motion.

(b) Show that by choosing suitable coordinates x and y in the frame of reference in which the planets are stationary, the lagrangian for the satellite may be written as

$$L = \frac{1}{2}m_3\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right] + Gm_3\left[\frac{m_1}{r_1} + \frac{m_2}{r_2}\right],$$

where you should determine r_1 and r_2 in terms of x, y, m_1, m_2, and r. [4]

[Unseen] We choose our co-ordinates such that the origin is at the centre-of-mass of the two planet system with both planets at $y = 0$. So planet one is at $(x, y) = (-r\mu/m_1, 0)$ and planet two is at $(x, y) = (+r\mu/m_2, 0)$. So if the satellite is at a general position (x, y), then its distance from planet one is (by Pythagoras) $r_1^2 = (x + r\mu/m_1)^2 + y^2$, while its distance from planet two is $r_2^2 = (x - r\mu/m_2)^2 + y^2$, where $\mu = \frac{m_1 m_2}{m_1+m_2}$ and these are simply the distances from the satellite to each of the planets. The potential energy is then given simply by the sum of the two contributions from each of the planets.

For the kinetic energy, we must remember that the reference frame is stationary with respect to the planets, so is rotating with angular velocity ω with respect to the (intertial) centre-of-mass-of-the-two-planets frame. In that frame, the velocity therefore has x component $\dot{x} - \omega y$ and y-component $\dot{x} + \omega x$ and the given lagrangian follows.

(c) Find the equation of motion for x and show that the equation of motion for y is given by

$$m_3(\ddot{y} + 2\omega\dot{x} - \omega^2 y) = -Gm_3\left[\frac{m_1 y}{r_1^3} + \frac{m_2 y}{r_2^3}\right].$$

[4]

[Partly seen] Using the standard formula $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$, the Euler-Lagrange equations are (after gathering terms)

$$m_3\ddot{y} = -2m_3\omega\dot{x} + m_3\omega^2 y - Gm_3\left[\frac{m_1 y}{r_1^3} + \frac{m_2 y}{r_2^3}\right]$$

and

$$m_3\ddot{x} = 2m_3\omega\dot{y} + m_3\omega^2 x - Gm_3\left[\frac{m_1(x + r\mu/m_1)}{r_1^3} + \frac{m_2(x - r\mu/m_2)}{r_2^3}\right].$$

(d) Find the locations of the two points away from the line joining the two planets at which the satellite may be stationary with respect to the planets. [4]

[Unseen] For stationary solutions, we set all time derivatives equal to zero. Since $y \neq 0$ by assumption, we can divide by y in the y equation giving

$$\omega^2 = G\left[\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right].$$

But plugging this into the term $m_3\omega^2 x$ in the x equation (with all time derivatives set to zero), we see that all terms in x cancel, leaving just

$$0 = r\mu/r_1^3 - r\mu/r_2^3,$$

so $r_1 = r_2$. Plugging this into

$$\omega^2 = G\frac{m_1 + m_2}{r^3} = G\left[\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right],$$

we see that $r_1 = r_2 = r$. So there are two solutions, in which the satellite forms the apex of an equilateral triangle with the two planets making the opposite side, both above and below the line joining the two planets.

(Students may also guess that the solution is $r_1 = r_2 = r$ and confirm this by plugging into the two equations.)

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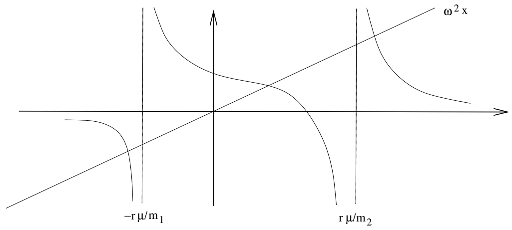
(e) By means of a graphical method, find out how many points there are on the line joining the two planets at which the satellite may be stationary with respect to the planets. [6]

[Unseen] Since $y = 0$, the E-L equation for y is trivially satisfied. For the x E-L equation, we have

$$\omega^2 x = G\left[\frac{m_1(x + r\mu/m_1)}{|x + r\mu/m_1|^3} + \frac{m_2(x - r\mu/m_2)}{|x - r\mu/m_2|^3}\right].$$

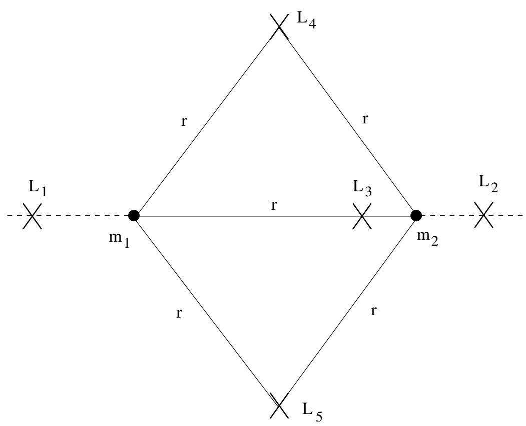
Plotting graphs of the left and right hand sides we obtain the plot shown (we have assumed $m_1 > m_2$ here, wlog). The plot is obtained as follows. For the right hand side,



we have the sum of two functions of the form $x/|x|^3$, but with their poles located at the positions of each of the two planets. Far away from either planet, the contribution of that planet is negligible. The functions are monotonically decreasing, so there are no stationary points and we get something of the form shown. So we see that there are precisely three solutions, located either side of the planets and between them.

(f) Draw a sketch showing the planets and all the possible stationary points. [2]

[Unseen] Putting together the information already obtained, we get



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2 A relativistic real scalar field ϕ in 1+1-dimensional spacetime, in units where $c = 1$ with co-ordinates $x^\mu = (t, x)$ and Minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, has lagrangian density given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi + m^2 \cos\phi,$$

where $m^2 > 0$.

(a) Write down the Euler-Lagrange equation of motion for ϕ. [2]

[Partly seen] Using the standard formula, we get

$$\partial^\mu \partial_\mu \phi + m^2 \sin\phi = 0.$$

(b) Write down the conserved energy-momentum tensor, in terms of ϕ and its derivatives, and discuss whether it is symmetric. [4]

[Partly seen] Using the standard formula, we have

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}.$$

We are not entitled to ask if this is symmetric, since it has indices of different type. But we may lower an index with the Minkowski metric $\eta_{\mu\nu}$ to get

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L},$$

which is manifestly symmetric. Explicitly, we get

$$\begin{aligned} T_{00} &= \frac{1}{2}(\partial_0 \phi \partial_0 \phi + \partial_1 \phi \partial_1 \phi) - m^2 \cos\phi \\ T_{01} &= \partial_0 \phi \partial_1 \phi = T_{10} \\ T_{11} &= \frac{1}{2}(\partial_0 \phi \partial_0 \phi + \partial_1 \phi \partial_1 \phi) + m^2 \cos\phi. \end{aligned}$$

(c) Find the values of α such that the transformation $\phi(x^\mu) \mapsto \phi(x^\mu) + \alpha$ is a symmetry; if there is a corresponding conserved current, find it. [2]

[Unseen] We need $\cos\phi$ to be invariant, so α must be an integer multiple of 2π. This is a discrete symmetry, so there is no conserved current.

(d) Show that there are no plane wave solutions to the Euler-Lagrange equation. [2]

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[Unseen] A plane wave solution of the form $Ce^{i(\omega t-kx)}$, would imply that

$$(\omega^2 - k^2)Ce^{i(\omega t-kx)} + m^2 \sin Ce^{i(\omega t-kx)} = 0.$$

Unlike in the case of the Klein-Gordon equation, and since $m^2 \neq 0$ by assumption, we cannot cancel the $e^{i(\omega t-kx)}$, so this cannot be a solution for all x, t, unless $C = 0$, in which case we don't have much of a wave.

(e) Show that a solution depending on x^μ only via the combination $\tau = t - x/v$, for some v such that $-1 < v < 1$, must satisfy the equation

$$\frac{d^2\phi}{d\tau^2} = A \sin\phi,$$

where A is a constant that you should determine in terms of m and v. [2]

[Unseen] For $\phi = \phi(t - x/v)$ we get $\partial^\mu\partial_\mu\phi = (\partial_t\partial_t - \partial_x\partial_x)\phi = (1 - 1/v^2)\frac{d^2\phi}{d\tau^2}$, using the chain rule. Rearranging, one finds $A = \frac{m^2v^2}{1-v^2}$.

(f) Find the values of B and C for which there is a solution of the form

$$\phi = 4 \arctan \exp{(B\tau + C)}$$

and draw a sketch of such a solution, explaining why it is physically reasonable. [8]

[Unseen] Starting with the left hand side, and using $e = \exp{(B\tau + C)}$ as shorthand, we get

$$\frac{d\phi}{d\tau} = \frac{4Be}{1+e^2}$$

and

$$\frac{d^2\phi}{d\tau^2} = \frac{4B^2e(1-e^2)}{(1+e^2)^2}.$$

As for the right hand side, we compute $\sin 4\mathrm{arctan}\, e$ as follows. Firstly we have that

$$\arctan e = \frac{i}{2}\log\frac{1-ie}{1+ie}$$

and that

$$\sin f = \frac{i}{2}(e^{-if} - e^{if}).$$

Putting these together, we get that

$$\sin 4\arctan e = \frac{i}{2}\exp\left(2\log\frac{1-ie}{1+ie}\right) - \frac{i}{2}\exp\left(2\log\frac{1+ie}{1-ie}\right),$$

or

$$\sin 4\arctan e = \frac{i}{2}\left(\frac{1-ie}{1+ie}\right)^2 - \frac{i}{2}\left(\frac{1+ie}{1-ie}\right)^2,$$

which simplifies to

$$\sin 4\arctan e = \frac{4e-4e^3}{(1+e^2)^2}.$$

Hence we get a solution if $B^2 = A = \frac{m^2v^2}{1-v^2}$, for any value of C.

(Students may also proceed using trigonometric formulae, as follows:

$$\begin{aligned}
\sin\phi &= \sin(4\arctan e)\\
&= 2\sin(2\arctan e)\cos(2\arctan e)\\
&= 4\sin(\arctan e)\cos(\arctan e)[2\cos^2(\arctan e) - 1]\\
&= 4\frac{e}{1+e^2}\left(\frac{2}{1+e^2} - 1\right)\\
&= -\frac{4}{e^{-1}+e} \times \frac{e-e^{-1}}{e+e^{-1}}.
\end{aligned}$$

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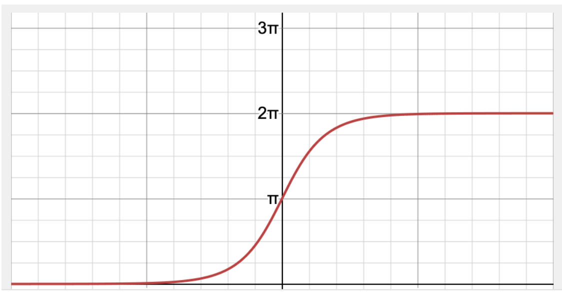
Here is a sketch (assuming $B > 0$ and $C = 0$): This solution interpolates between



$\phi = 0$ at $\tau \to -\infty$ and $\phi = 2\pi$ at $\tau \to +\infty$. It is smooth and its derivatives go to zero at $\pm\infty$, so we get finite kinetic energy. It goes to zeros of the potential energy as $\pm\infty$ so we get finite total energy.

(g) Identify as many symmetries of the system as you can and discuss whether these can be used to find new solutions from the solutions you have already found. [5]

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[Unseen] A first observation is that the invariance of the action under space or time translations does not give new solutions, since it corresponds to shifting C, and we have already seen that any C yields a solution. Similarly, the lagrangian is invariant under $\tau \mapsto -\tau$, i.e. $(t, x) \mapsto -(t, x)$, but this is already captured by the fact that a solution remains a solution when we flip the sign of B. The lagrangian is invariant under $t \mapsto -t$ alone, but this is already captured by the fact that a solution remains a solution when we flip the sign of v. The lagrangian is also invariant under Lorentz boosts, but these again simply shift the value of v, generating all values with $|v| < 1 = c$.

However, we have already seen that there is a symmetry when we shift ϕ by an integer multiple of 2π, and this does give more solutions, which interpolate not between the potential minima at 0 and 2π but rather between any minimum and the next one. Finally, the lagrangian is also invariant under $\phi \mapsto -\phi$, but this has the same effect as reversing τ and shifting by 2π.

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3 Consider a particle moving in $2+1$ spacetime dimensions described by a hamiltonian H. The differential equation describing the propagation of this particle from position $\boldsymbol{r}$ at time t to position $\boldsymbol{r'}$ at time t' is given in terms of the Green function $G(\boldsymbol{r}, \boldsymbol{r'}, t, t')$ by

$$\left(i\hbar\frac{\partial}{\partial t} - H\right) G(\boldsymbol{r}, \boldsymbol{r'}, t, t') = \delta^2(\boldsymbol{r} - \boldsymbol{r'})\delta(t - t').$$

The Green function is to be studied using the Fourier transforms

$$G(\boldsymbol{r}, \boldsymbol{r'}, E) = \int dt\ \exp(iE(t-t')/\hbar) G(\boldsymbol{r}, \boldsymbol{r'}, t, t'),$$

$$G(\boldsymbol{k}, E) = \iint d^2\boldsymbol{r}\ \exp(i\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r'})) G(\boldsymbol{r}, \boldsymbol{r'}, E).$$

(a) Describe what it means for a Green function to be causal and how causality may be implemented using Fourier transforms. [5]

[Seen] A Green function is said to be causal if it vanishes for $t < t'$ (for all values of $\boldsymbol{r}$ and $\boldsymbol{r'}$) and corresponds to a choice of initial conditions in the differential equation. Clearly this is desirable on physical grounds. If we write the Green function in terms of an integral over E, i.e. in terms of the inverse Fourier transform of the first one given above, as

$$G(\boldsymbol{r}, \boldsymbol{r'}, t, t') = \int_{\infty}^{\infty} \frac{dE}{2\pi\hbar}\ \exp(-iE(t-t')/\hbar) G(\boldsymbol{r}, \boldsymbol{r'}, E),$$

the choice of initial conditions for the Green function in t corresponds to a freedom to choose how we go around the singularities on the real axis in the complex E plane in the integral. To get the causal boundary condition we should go above the poles, since for

$t < t'$, we can close the integration contour by means of a semi-circle in the upper half plane whose contribution to the integral becomes negligible as the semi-circle becomes large. Thus, since no singularities are enclosed, the original integral is zero by Cauchy's theorem and we reproduce the causal boundary condition. Equivalently, we can give the energy a small positive imaginary part.

(b) Suppose that the particle is of mass m and propagates non-relativistically and freely, such that the Hamiltonian is $H = -\frac{\hbar^2\nabla^2}{2m}$. Show that, with causality, the Green function $G(\boldsymbol{r}, \boldsymbol{r'}, E)$ can be written, for $\boldsymbol{r'} = \boldsymbol{r}$ and $E > 0$, as

$$G(\boldsymbol{r}, \boldsymbol{r'} = \boldsymbol{r}, E > 0) = \lim_{\delta \to 0} G_\delta,$$

where

$$G_\delta = \alpha \int_0^\infty dk \left(\frac{1}{k - k_+ - i\delta} + \frac{1}{k + k_+ + i\delta} \right), \tag{1}$$

δ is real, positive, and arbitrarily small, and α and k_+ are constants that you should determine. [7]

[Partly seen] Taking the differential equation and applying $\int dt e^{iE(t-t')/\hbar}$ to both sides yields

$$(E - H)G(\boldsymbol{r}, \boldsymbol{r}', E) = \delta^2(\boldsymbol{r} - \boldsymbol{r'}),$$

where in the first term on the LHS we used integration by parts. Next we apply $\iint d^2\boldsymbol{r} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$ to both sides to get

$$\left(E - \frac{\hbar^2 k^2}{2m}\right) G(\boldsymbol{k}, E) = 1,$$

where $k = |\boldsymbol{k}|$. Taking the inverse Fourier transform, we now get

$$G(\boldsymbol{r}, \boldsymbol{r}', E) = \frac{1}{4\pi^2} \iint d^2\boldsymbol{k} e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \frac{1}{\left(E - \frac{\hbar^2 k^2}{2m}\right)}.$$

We want

$$G(\boldsymbol{r}, \boldsymbol{r}, E > 0) = \frac{1}{4\pi^2} \int d^2\boldsymbol{k} \frac{1}{\left(E - \frac{\hbar^2 k^2}{2m}\right)} = \frac{-m}{\pi\hbar^2} \int_0^\infty dk \frac{k}{k^2 - k_+^2} = \frac{-m}{2\pi\hbar^2} \int_0^\infty dk \left[\frac{1}{k - k_+} + \frac{1}{k + k_+}\right],$$

where we used plane polar co-ordinates and where $k_+^2 = 2mE/\hbar^2$. Let us now assume that $E > 0$ as the question instructs us, and take k_+ to be the positive root of $2mE/\hbar^2$. We have argued that causality is enforced by adding a small positive imaginary part, ϵ say, to E, such that the poles get shifted to $\pm(k_+ + i\delta)$ where $\delta \propto \epsilon$ is also real, positive, and small. So we end up with

$$G(\boldsymbol{r}, \boldsymbol{r}, E > 0) = \lim_{\delta \to 0} \frac{-m}{2\pi\hbar^2} \int_0^\infty dk \left[\frac{1}{k - k_+ - i\delta} + \frac{1}{k + k_+ + i\delta}\right],$$

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whence we read off that $k_+ = +\sqrt{2mE}/\hbar$ and $\alpha = \frac{-m}{2\pi\hbar^2}$.

(c) Use the formula

$$\rho(E) = \frac{-1}{2\pi i} \lim_{\delta\to 0} \left[G_{|\delta|} - G_{-|\delta|}\right],$$

where G_δ is defined for both positive and negative values of δ by equation (1) above, to compute the density of states $\rho(E)$ for $E > 0$. [7]

[Unseen] Inspecting the form of G_δ given, we see immediately that $G_{|\delta|} - G_{-|\delta|} = 2i\mathrm{Im}G_{|\delta|}$. So now we can use the hint to write

$$\rho(E) = \frac{-2i}{2\pi i} \lim_{\delta\to 0} \mathrm{Im}G_{|\delta|} = \frac{m}{2\pi^2\hbar^2}\mathrm{Im}\int_0^\infty dk\left[i\pi\delta(k-k^+) - i\pi\delta(k+k^+)\right] = \frac{m}{2\pi\hbar^2}(1+0) = \frac{m}{2\pi\hbar^2},$$

so the density of states is constant.

(d) In graphene, the hamiltonian can be written in terms of $\boldsymbol{k}$ as $H = \hbar v|\boldsymbol{k}|$, where v is a real positive constant. Calculate the density of states $\rho(E)$ for $E > 0$. [6]

[Unseen] Following the same steps in the calculation as before, we get

$$\rho(E) = \frac{-1}{\pi} \lim_{\delta\to 0} \mathrm{Im}G_{|\delta|} = \frac{-1}{\pi}\frac{-1}{\hbar v 2\pi}\mathrm{Im}\int_0^\infty k\pi i\delta(k-k_0) = \frac{E}{2\pi\hbar^2 v^2},$$

where $k_0 = E/\hbar v$. The expression for G_δ corresponding to equation (1) above is

$$G_\delta = -\frac{1}{2\pi\hbar v}\int_0^\infty \frac{kdk}{k-k_0-i\delta}.$$

$\Big[$*Hint: For real x and real positive y, the imaginary part of* $\lim_{y\to 0}\frac{1}{x+iy}$ *is given by* $-\pi\delta(x)$.$\Big]$

4 This question involves using a microscopic model to study phase transitions in a system in thermal equilibrium at temperature T.

(a) Describe the meaning of the term *order parameter* in the context of phase transitions and give an example of such an order parameter. [4]

[Bookwork] The order parameter is a macroscopic quantity that characterizes in general the phases of a system in thermodynamic equilibrium and measures in particular the extent to which symmetries are broken in phases. Typically it vanishes in the disordered phase and is non-vanishing in the ordered phase; an example is the magnetisation in a ferromagnet.

The Dicke model describes the interaction of light, modelled by a real-valued field α, with a number N of two-state atomic emitters. The partition function of the model is given by

$$Z(\alpha) = e^{-\beta\omega\alpha^2} \left(\mathrm{Tr}\, e^{-\beta h(\alpha)}\right)^N .$$

Here, $h(\alpha)$ is the 2×2-matrix given by

$$h(\alpha) = \frac{\Delta}{2}\sigma_z + \frac{2g}{\sqrt{N}}\alpha\sigma_x,$$

while $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the usual Pauli matrices, $\beta = 1/kT$, and k, ω, Δ, and g are real positive constants.

(b) Show that the free energy, $F(\alpha) = -\frac{1}{\beta}\ln\left(Z(\alpha)\right)$, can be written as

$$F(\alpha) = b\alpha^2 - c\ln\Big(2\cosh(\beta E(\alpha))\Big),$$

where you should determine b, c and $E(\alpha)$ in terms of the parameters already given. [7]

[Partly Seen] To calculate the trace, it is easiest to first diagonalise $h(\alpha)$, for which we find

$$h(\alpha) = \begin{pmatrix} E_+(\alpha) & 0 \\ 0 & E_-(\alpha) \end{pmatrix},$$

where $E_\pm(\alpha) = \pm\sqrt{\frac{\Delta^2}{4} + \frac{4\alpha^2 g^2}{N}}$. Therefore, for Z we find

$$Z = \exp(-\beta\omega\alpha^2)\left[2\cosh(\beta E_+(\alpha))\right]^N .$$

Using the given expression for the free energy we find that

$$F(\alpha) = \omega\alpha^2 - \frac{N}{\beta}\ln\Big(2\cosh(\beta E(\alpha))\Big),$$

so $b = \omega$, $c = N/\beta$ and $E(\alpha) = E_+(\alpha) = \sqrt{\frac{\Delta^2}{4} + \frac{4\alpha^2 g^2}{N}}$.

(c) Explain why the free energy always has either one minimum at $\alpha = 0$ or two minima, each at non-zero values of α (you need not find the explicit values of α). [7]

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[Partly Seen] Let us first try to find the stationary points the free energy, i.e. to solve

$$F'(\alpha) = 0.$$

Explicitly, we obtain

$$2\omega\alpha - \frac{4\alpha g^2}{E(\alpha)} \tanh\left(\beta E(\alpha)\right) = 0.$$

Here $\alpha = 0$ is obviously one solution (which would, if it is the global minimum of the free energy, describe a disordered phase.) Stationary points with non-zero values of α (which would correspond to ordered phases) correspond to solutions of

$$2\omega - \frac{4g^2}{E(\alpha)} \tanh\left(\beta E(\alpha)\right) = 0.$$

Now, considered as an equation in $E(\alpha) \in \mathbb{R}$, this certainly has a solution at $E(\alpha) = 0$, but this does not correspond to a real value of α. For certain values of the couplings, namely when $\omega/2\beta g^2 < 1$ we get two more solutions in $E(\alpha)$, which in turn only correspond to real values of α if they also satisfy $|E(\alpha)| > \Delta/2$. Due to the symmetry of the free energy and $E(\alpha)$ under $\alpha \mapsto -\alpha$, we only ever get two solutions for α in this way, which are equal and opposite in sign. So in sum we only every get one or three stationary points.

Now we must address the question of which of these stationary points are minima. Since the free energy is easily seen to be bounded below (recall that $\omega > 0$), it must have a (global) minimum somewhere. Moreover, since it is, smooth, symmetric under $\alpha \mapsto -\alpha$, and has at most three stationary points, simple considerations of continuity show that we either have a minimum at $\alpha = 0$ (and either two degenerate points of inflection or no other stationary points elsewhere) or we have two degenerate minima at equal and opposite values of $\alpha \neq 0$, in which case we must have a maximum at $\alpha = 0$. So we either have one minima at $\alpha = 0$ or one maxima at $\alpha = 0$ and two degenerate minima (at equal and opposite non-zero values of α).

(d) Show that, when $g^2/\omega\Delta$ exceeds a certain value (which you should determine), there is a phase transition and find the temperature at which the transition occurs. [7]

[Unseen] We have already shown that there is either a minimum at $\alpha = 0$ or a maximum at $\alpha = 0$, in which case there are degenerate minima at equal and opposite non-zero values of α. This gives a simple criterion for when the phase transition occurs, even though we cannot solve explicitly for α, namely when the stationary point at $\alpha = 0$ switches from being a minimum to a maximum. And this occurs when the second derivative of the free energy vanishes at $\alpha = 0$.

The second derivative is not pleasant, but we may discard terms proportional to α, since they evaluate to 0 at $\alpha = 0$. So we get

$$F''(\alpha) = 2\omega - \frac{4g^2}{E(\alpha)}\tanh\beta E(\alpha) + O(\alpha),$$

which at $\alpha = 0$ is given by

$$F''(0) = 2\omega - \frac{8g^2}{\Delta}\tanh\frac{\beta\Delta}{2}.$$

This equation has a solution if and only if $\frac{\Delta\omega}{4g^2} < 1$ (since the hyperbolic tangent cannot exceed unity) or $\frac{g^2}{\omega\Delta} > \frac{1}{4}$.

If this condition is satisfied, we get a phase transition, with critical temperature given by

$$\beta_c = \frac{2}{\Delta}\operatorname{arctanh}\frac{\Delta\omega}{4g^2}$$

or, in other letters,

$$T_c = \frac{\Delta}{2k\operatorname{arctanh}\frac{\Delta\omega}{4g^2}}.$$

(Students may also solve this by observing that solutions with $\alpha \neq 0$ first appear when $E(\alpha) > \Delta/2$ and plugging this condition into the equation $F'(\alpha) = 0$, but they must also explain why these solutions are minima, rather than points of inflection.)

END OF PAPER